Differential method for wavefront sensor measurements of turbulence parameters and wind velocity

L.V. Antoshkin, V.V. Lavrinov, L.N. Lavrinova, and V.P. Lukin

Institute of Atmospheric Optics,
Siberian Branch of the Russian Academy of Sciences, Tomsk

Received August 19, 2007

The theoretical basis for applying the differential method and the correlation analysis for measuring the structural constant of refractive index, coherence length, and cross component of wind velocity is presented. A wavefront sensor is considered as the measuring instrument. The theoretical results are confirmed by numerical experiments.

Propagation of optical radiation in the atmosphere is accompanied with fluctuations of its parameters (intensity, phase, angle of arrival, etc.). Ray path changes caused by atmospheric turbulence result in phase fluctuations along and across the beam. Phase fluctuation along the beam decreases time coherence, while the fluctuations across the beam disturb the spatial wavefront coherence, distorting and curving the beam and causing image "jitter." Phase fluctuations result in fluctuations of angle of arrival. Estimation of the latter fluctuations on the base of the cross correlation analysis allows one to determine characteristic parameters of the atmospheric turbulence, while the use of differential method – to minimize the measurement errors caused by proper oscillations of the measuring system.

The idea of using measurements of the angle of stellar light arrival in order to determine turbulence parameters was proposed in the middle of the last century.[1] Numerical relations between angles of arrival and turbulence parameters were defined by Fried.[2] Differential measurements of stellar light movement were realized in the Differential Image Motion Monitor,[3] which calculates turbulence parameters from the dispersion of random shifts of image energy centers. In general, the differential monitor is an effective and available tool, capable of measuring several turbulence parameters; at present, it is used at such observatories as Cerro Paranal or Mauna-Kea. At the same time, it cannot provide for total information on detailed turbulence structure.

The ideology of Grating Scale Monitor (GSM)[4] is based on the analysis of spatial covariation of arrivel angle fluctuations within the Karman model. The monitor uses the principle similar to the Shack–Hartmann sensor, i.e., it measures fluctuations of the arrival angle, detectable simultaneously at several wavefront points, and can provide for almost complete set of wavefront parameters, important for methods with high angular resolution. To measure wavefront velocity, GSM considers the temporal cross correlation of the arrival angles between two telescopes spaced by a fix distance. Application of the differential method in GSM allows to eliminate the influence of noise sources on visibility detection.[5]

The essence of the differential method, used as the basis of the differential turbulence meter,[6] consists in calculating turbulence parameters from the measured difference dispersion of angular shifts α_1 and α_2 of energy centers of images from two D-diameter subapertures, located in the entrance pupil plane at the distance d, ignoring the anisotropy ($\sigma^2_{\alpha_1} = \sigma^2_{\alpha_2}$), by the equation

$$\sigma^2_\alpha = 2r_0^{-5/3}\lambda^2 q, \quad (1)$$

which allows calculation of the path-averaged structural constant of refraction index

$$C_n^2 = \frac{\sigma^2_\alpha}{3.384\pi^2 Lq} \quad (2)$$

and the Fried radius of a plane wave

$$r_0 = \left(\frac{\sigma^2_\alpha}{2\lambda^2 q}\right)^{-3/5}, \quad (3)$$

where $\sigma^2_\alpha = \sigma^2_{\alpha_1-\alpha_2}$; $q = A_\alpha D^{-1/3} - Fd^{-1/3}$ is the constant, characterizing the applicability of the differential method, $F = 0.097$ for longitudinal correlation and $F = 0.145$ for the cross one, $A_\alpha = \dfrac{A}{1.692\pi^2}$,

$$A = 1.46 \text{ for } l_0 < D < \sqrt{\lambda L} \text{ and } A = 2.9 \text{ for } L_0 > D > \sqrt{\lambda L} \quad (4)$$

within the Kolmogorov turbulence model, L is the path length; $[l_0, L_0]$ is the inertial interval of spatial scale inhomogeneities.

The differential method for minimizing the calculation error caused by proper oscillations of the measuring system allows structural turbulence characteristics to be determined.

0235-6880/08/01 64-05 $02.00

The characteristic turbulence parameters can be determined with a wavefront sensor as well. Figure 1 shows the schematic of a wavefront sensor (WFS) of the Shack–Hartmann type. The lens raster divides the arriving wavefront to local areas, which are focused on the receiver, where then images are formed (hartmanograms).

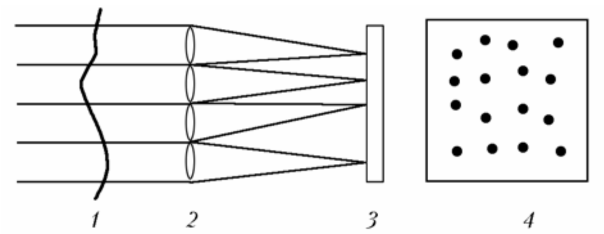

Fig. 1. Shack–Hartman WFS schematic: wavefront (*1*), lens raster (*2*), receiver (*3*), and hartmanogram (*4*).

The principle of operation of the Shack–Hartman WFS uses the measuring of local wave tilts with the coordinates

$$x_k = \sum_{i,j} x_{i,j} I_{i,j} / \sum_{i,j} I_{i,j}, \quad y_k = \sum_{i,j} y_{i,j} I_{i,j} / \sum_{i,j} I_{i,j},$$

expressed in radians via the receiver image scale; $I_{i,j}$ is the light intensity on receiver pixels; i and j are the pixel numbers in the focal area under measurement. Local wave tilts are proportional to ECSs of a focal spot with coordinates (x_k, y_k) relative to the spot obtained for a plane wavefront with coordinates (x_k^0, y_k^0):

$$\frac{\partial\varphi(x_k, y_k)}{\partial x} = \frac{1}{f}(x_k^0 - x_k), \quad \frac{\partial\varphi(x_k, y_k)}{\partial y} = \frac{1}{f}(y_k^0 - y_k), \quad (5)$$

where f is the microlens focus.

Transmission of optical radiation through a single microlens BC with diameter D is shown in Fig. 2.

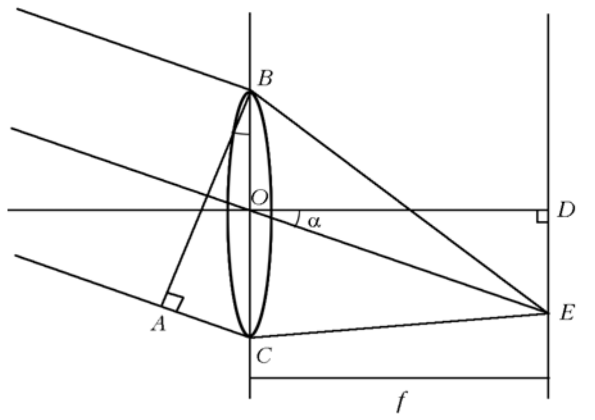

Fig. 2. Schematic of optical beam refraction by a single microlens.

The straight line DE corresponds to the receiver plane, and AB – to the plane of arriving wavefront. The segment DE is equal to the focal spot shift $x^0 - x$ and proportional to the angle of arrival α. The angle of wavefront tilt relative to the microlens plane $\angle ABC$ is equal to α and proportional to the segment AC, corresponding to the phase difference $\varphi(A) - \varphi(B)$.

Thus, $\tan\alpha = (x^0 - x)/f$ can be determined from the shift of calculated coordinates of the focal spot (as at small α $\tan\alpha \approx \alpha$, then $\alpha = (x^0 - x)/f$), as well as the tilt angle and the phase difference $\Delta\varphi = [D(x^0 - x)/f]$ (taking into account that $\sin\alpha \approx \alpha$ for small α).

On the one hand, the dispersion of the angle of arrival can be calculated as

$$\sigma_\alpha^2 = \sum_{m=1}^{M} \frac{(\alpha_m - \bar{\alpha})^2}{M - 1}, \quad (6)$$

where M is the number of measurements of the angle of arrival in the focal spot; $\bar{\alpha} = \langle \alpha_m \rangle$ is the path-averaged angle of arrival.

In the coordinate system x, y, the angle of arrival depends on the distance $d = \sqrt{(x^0 - x)^2 + (y^0 - y)^2}$ and has the form $\alpha = d/f$; therefore, its dispersion can be expressed via the dispersion of ECS of the focal spot:

$$\sigma_\alpha^2 = \frac{1}{f^2} \sum_{m=1}^{M} \frac{(d_m - \bar{d})^2}{M - 1}, \text{ or } \sigma_\alpha^2 = \frac{1}{f^2} \sigma_d^2, \quad (7)$$

where $d_m = \sqrt{(x_m^0 - x_m)^2 + (y_m^0 - y_m)^2}$ is the ECS of the focal spot; $\bar{d} = \langle d_m \rangle$ is the path-averaged ECS of the same focal spot.

On the other hand, the dispersion of angle of arrival within the Kolmogorov turbulence model[7] has the form

$$\sigma_\alpha^2 = \begin{cases} 1.46 D^{-1/3} C_n^2 L & \text{for } l_0 < D < \sqrt{\lambda L} \\ 2.9 D^{-1/3} C_n^2 L & \text{for } L_0 > D > \sqrt{\lambda L}, \end{cases} \quad (8)$$

or

$$\sigma_\alpha^2 = A L C_n^2 D^{-1/3}$$

on conditions (4). Equating the right parts of Eqs. (7) and (8), define C_n^2 via the dispersion of the focal spot ECS:

$$C_n^2 = \frac{D^{1/3}}{A f^2 L} \sum_{m=1}^{M} \frac{(d_m - \bar{d})^2}{M - 1}, \text{ or } C_n^2 = \frac{D^{1/3}}{A f^2 L} \sigma_d^2. \quad (9)$$

As in a differential turbulence meter, the differential method and cross correlation signal analysis[6] are applicable for the wavefront sensor when calculating turbulence parameters from the measured shift-difference dispersion of the focal spot energy centers of the hartmanogram, resulting from optical radiation transmission though the lens raster.

The shift-difference dispersion of energy centers is calculated for pairs of focal spots, corresponding to a pair of microlens of D in diameter with the distance d_{mc} between their centers and distance d between neighboring microlenses. The relation between the microlenses' centers

$$d_{mc} = \sqrt{(\xi_{ij} - \xi_{i+kj+l})^2 + (\eta_{ij} - \eta_{i+kj+l})^2}, \quad \xi_{ij} = \frac{x_{ij}^0}{h},$$

$$\eta_{ij} = \frac{y_{ij}^0}{h}, \quad \xi_{i+kj+l} = \frac{x_{i+kj+l}^0}{h}, \quad \eta_{i+kj+l} = \frac{y_{i+kj+l}^0}{h}$$

and the distance between centers of corresponding focal areas

$$d_f = \sqrt{(x_{ij}^0 - x_{i+kj+l}^0)^2 + (y_{ij}^0 - y_{i+kj+l}^0)^2}, \quad d_f / d_{mc} = h$$

depends on the camera resolution: $h = D_z/D$, where D_z is the size of focal area in the hartmanogram.

The path-averaged structural constant is calculated, according to Eq. (2), from the measured shift-difference dispersion of the energy centers of a pair of focal spots with the distance d_f between their centers as

$$C_n^2 = \frac{\sigma_d^2}{3.384\pi^2 f^2 L q_{mc}}, \tag{10}$$

where $q_{mc} = A_\alpha D^{-1/3} - F d_{mc}^{-1/3}$, d_{mc} is the distance between the centers of chosen microlenses; $A_\alpha = A/1.692\pi^2$ under conditions (4); the shift-difference dispersion of the energy centers of focal spots has the form

$$\sigma_d^2 = \sum_{m=1}^{M} \frac{[(d_{ij} - d_{i+kj+l})_m - (\overline{d_{ij} - d_{i+kj+l}})^2]}{M-1}, \tag{11}$$

where

$$d_{ij} = \sqrt{(x_{ij}^0 - x_{ij})^2 + (y_{ij}^0 - y_{ij})^2},$$

$$d_{i+kj+l} = \sqrt{(x_{i+kj+l}^0 - x_{i+k+lj})^2 + (y_{+kij+l}^0 - y_{i+kj+l})^2}$$

are the ECSs of focal spots with coordinates (x_{ij}, y_{ij}), (x_{i+kj+l}, y_{i+kj+l}) relative to focal spots with coordinates (x_{ij}^0, y_{ij}^0), $(x_{i+kj+l}^0, y_{i+kj+l}^0)$, respectively: $i = 1, 2, ..., N$; $j = 1, 2, ..., N$; $k = 1, 2, ..., N-1$; $l = 1, 2, ..., N-1$.

When measuring the shift-difference dispersion of the energy centers of a pair of neighboring focal spots at $d_{mc} = d = D$, then $q_{mc} = D^{-1/3}(A_\alpha - F) = D$, or $q_{mc} = D^{-1/3}(A_\alpha - F\sqrt{2})$. In this case, the difference method reduces to Eq. (9) and C_n^2 is calculated by the dispersion of the ECSs of one focal spot.

The application of cross correlation analysis to the work of a wavefront sensor allows calculating the average cross component of the wind velocity. As is known,[8] according to "frozeness" hypothesis, the turbulence "swims" under the action of the cross component of wind velocity in a plane, parallel to the lens raster plane, and, hence, hartmanogram plane (see Fig. 1).

Figure 3 shows a virtual coincidence of the turbulence movement plane under the action of wind velocity cross component with the hartmanogram plane.

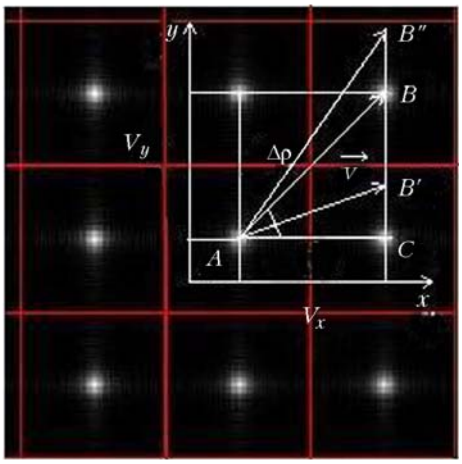

Fig. 3. Hartmanogram fragment: V_x, V_y are the wind velocity projections.

The x- and y-components of cross wind velocity, in terms of which energy center shifts (ECS) of the focal spot are defined, evidently form the angle $\theta = \arctan(V_y/V_x)$. In general, the velocity varies linearly: $V = \text{k}\Delta\rho/\Delta t$, $\text{k} = V_y/V_x$. In this case, the correlation time is $\Delta t = t_1 - t_m$, where t_1 and t_m are the time points provided the coordinates of energy centers of focal spots are equal, i.e., $x_k(t_1) = x_k(t_m)$ and $y_k(t_1) = y_k(t_m)$, since the turbulence movement within the Kolmogorov model is a periodic process.[9]

According to Eqs. (1) and (7), the shift-difference dispersion of energy centers of a pair of focal spots has the form

$$\sigma_d^2 = 2 f^2 r_0^{-5/3} \lambda^2 (A_\alpha D^{-1/3} - F d_{mc}^{-1/3}). \tag{12}$$

The differential method is unrealizable for $d_{mc} = d$. The physical sense of the method is in the value of difference $A_\alpha D^{-1/3} - F d_{mc}^{-1/3}$, or, more precisely, in the numerical relation between D and d_{mc}. The method is sufficiently well realizable if $d_{mc}^{-1/3}$ is vanishing or d_{mc} is quite large, i.e., the distance between the centers of chosen microlenses essentially exceeds the microlens' diameters. The parameters i, j, k, and l can be chosen so that d_{mc} corresponds to maximally distant microlenses, i.e., mounted at the raster edges.

At the same time, the use of correlation analysis in calculation of turbulence parameters by the shift-difference dispersion of energy centers of a pair of focal spots is possible only under the condition that fluctuations of focal spot ECSs are non-correlated. For this, the distance d_{mc} between the centers of chosen lenses should be larger than the outer scale of turbulence L_0.

If $d_{mc} \neq d$ or, in other words, for $A_\alpha D^{-1/3} - F d_{mc}^{-1/3}$, where $d_{mc} = (N_{\text{lens}} - 1)D$, N_{lens} is the lens raster size for $L_0 > D > \sqrt{\lambda L}$, taking into account $d_f / d_{mc} = h$, the distance d_f between the centers of corresponding focal areas can be expressed from Eq. (12) as follows:

$$d_f = h\left(\frac{A_\alpha D^{-1/3}}{F} - \frac{\sigma_d^2}{2f^2 r_0^{-5/3}\lambda^2 F}\right)^{-3}, \tag{13}$$

or

$$d_f = \frac{h}{(\pi^2 F)^{-1/3}}\left(\frac{AD^{-1/3}}{1.692} - \frac{\sigma_d^2}{3.384 f^2 C_n^2 L}\right)^{-3}. \tag{14}$$

It follows from Fig. 3, that

$$\Delta\rho = \frac{d_f}{\cos\theta} \text{ and } V = \frac{\Delta\rho}{\Delta t} = \frac{d_f}{\Delta t\cos\theta},$$

then

$$d_f = V\Delta t\cos\theta. \tag{15}$$

Equating the right parts of Eqs. (14) and (15), obtain the equation for the cross component of wind velocity

$$V = \frac{d_f}{\Delta t\cos\theta(1.692\pi^2 F)^{-1/3} d_{mc}}\left(AD^{-1/3} - \frac{\sigma_d^2}{2f^2 C_n^2 L}\right)^{-3}, \tag{16}$$

or

$$V = \frac{d_f}{\Delta t\cos\theta A_d d_{mc}}\left(AD^{-1/3} - \frac{\sigma_d^2}{2f^2 L C_n^2}\right)^{-3}, \tag{17}$$

where $A_d = 0.391F^{-1/3}$; $A = 0.29$.

In Eq. (17), the difference $A_\alpha D^{-1/3} - F d_{mc}^{-1/3}$ is absent in an explicit form and the velocity depends on the ratio σ_d^2 / C_n^2. This difference is included in the equation for structural constant C_n^2, and, hence the differential method is used in calculation of the wind velocity cross component.

This component is projected on the coordinate axes (see Fig. 3). The projections form an angle, which varies with time. In Eq. (17), wind direction variations are characterized by the $\cos\theta$.

Let the point A (see Fig. 3) has coordinates (i, j), point C – $(i + k, j)$, and point B – $(i + k, j + l)$; d_{fx} is the distance between points A and C, d_{fy} is the distance BC. Then $\tan\theta = V_y / V_x$ at every moment of time is determined by the ratio d_{fy}/d_{fx}, which (according Eq. (13) depends on

$$\sigma_{dx}^2 = \sum_{m=1}^{M}\frac{[(d_{i+kj} - d_{ij})_m - (\overline{d_{i+kj} - d_{ij}})^2]}{M-1}$$

and

$$\sigma_{dy}^2 = \sum_{m=1}^{M}\frac{[(d_{i+kj+l} - d_{i+kj})_m - (\overline{d_{i+kj+l} - d_{i+kj}})^2]}{M-1}.$$

The dependence of the angle between the velocity projections on the shift-difference dispersion of the energy centers of pairs of focal spots is shown in Fig. 4.

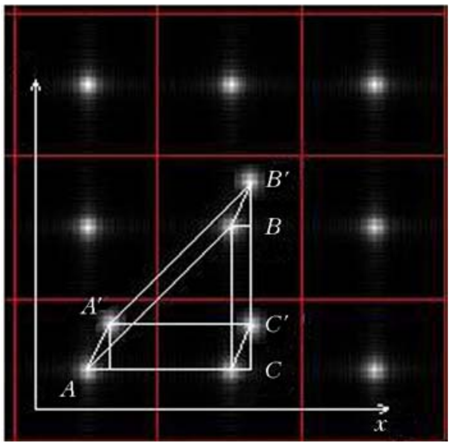

Fig. 4. Hartmanogram fragment: $V_x = AC$, $V_y = BC$ and $V'_x = A'C'$, $V'_y = B'C'$ are the velocity projections at the time moments t_1 and t_2.

At the initial moment t_1,

$$\theta = \arctan(V_y / V_x),$$

$$V_x = (x_C - x_A)^2 + (y_C - y_A)^2 = (x_{i+kj}^0 - x_{ij}^0)^2 + (y_{i+kj}^0 - y_{ij}^0)^2$$

corresponds to the distance d_{fx} between points A and C,

$$V_y = (x_B - x_C)^2 + (y_B - y_C)^2 = (x_{i+kj+l}^0 - x_{i+kj}^0)^2 + (y_{i+kj+l}^0 - y_{i+kj}^0)^2$$

corresponds to d_{fy} (distance BC). At the next moment t_2,

$$\theta = \arctan(V'_y / V'_x),$$

where the velocity projections are proportional to the shift-difference dispersion of the energy centers of pairs of focal spots, which can be analytically expressed as

$$V'_x = V_x + [(x_{i+kj} - x_{i+kj}^0)^2 + (y_{i+kj} - y_{i+kj}^0)^2] - $$
$$-[(x_{ij} - x_{ij}^0)^2 + (y_{ij} - y_{ij}^0)^2] =$$
$$= V_x + d_{i+kj} - d_{ij} = d_{fx} + d_{i+kj} - d_{ij},$$

$$V'_y = V_y + [(y_{i+kj+l} - y_{i+kj+l}^0)^2 + (y_{i+kj+l} - y_{i+kj+l}^0)^2] -$$
$$-[(x_{i+kj} - x_{i+kj}^0)^2 + (y_{i+kj} - y_{i+kj}^0)^2] =$$
$$= V_y + d_{i+kj+l} - d_{i+kj} = d_{fy} + d_{i+kj+l} - d_{i+kj},$$

$$(x_{ij}^0, y_{ij}^0),\ (x_{i+kj}^0, y_{i+kj}^0),\ (x_{i+kj+l}^0, y_{i+kj+l}^0)$$

are focal spot coordinates at the time t_1;

$$(x_{ij}, y_{ij}),\ (x_{i+kj}, y_{i+kj}),\ (x_{i+kj+l}, y_{i+kj+l})$$

are focal spot coordinates at the time t_2; $i = 1, 2, ..., N$; $j = 1, 2, ..., N$; $k = 1, 2, ..., N - 1$; $l = 1, 2, ..., N - 1$.

At the moment t_m,

$$V_x^{(m)} = V_x + \sum_{m=1}^{M} \frac{(d_{i+kj} - d_{ij})_m}{M-1},$$

$$V_y^{(m)} = V_y + \sum_{m=1}^{M} \frac{(d_{i+k+lj} - d_{i+kj})_m}{M-1}.$$

The path-averaged wind velocity is determined by the shift-difference dispersion of the energy centers of pairs of focal spots σ_{dx}^2 and σ_{dy}^2 centered at the points A, B, and C, respectively.

The use of a CCD-camera as a receiver allows measuring the dispersion of angle of arrival, which is usually 1–10″. The focal spot diameter D_f depends on the CCD-camera resolution, microlens diameter, and focus. If the camera resolution is 512×512 pixels, then the focal area is 64×64 pixels for the lens raster of 24 mm in size, consisting of 64 microlenses (8×8). The pixel size corresponds to 10×10 μm. Thus, the focal area in this case is 640×640 μm and corresponds to a microlens of 3000×3000 μm in size. Random shifts of a focal spot should not exceed 1/3 of the radius of diffraction pattern to avoid the focal spot expansion beyond its zone. The focal spot radius corresponds to the radius of the third dark fringe in the Airy diffraction pattern $D_f = 1.619\lambda/D$. The maximum allowable angular ECS of the focal spot is $8.5 \cdot 10^{-5}$ rad for a microlens with a diameter of 3 mm. The dispersion of angular shifts should not exceed $7.2 \cdot 10^{-10}$ rad. If α is the maximum allowable angular ECS of the focal spot, then the focal distance $f = D_f/2\alpha = 1.619\lambda/2\alpha D$ answers it. In our case, $f \approx 2$ m for $\lambda = 0.63$ μm, hence, $\alpha = 1.7 \cdot 10^{-11}$ m.

To estimate the efficiency of turbulence parameter measurements with a wavefront sensor, a numerical model has been built, which includes the dynamic turbulence model with the Karman inhomodeneities,[8,9] lens raster, and Shack–Hartmann sensor model. The value of abscissa shift of focal spot energy center was obtained from numerical experiments; it equals to $1.51765 \cdot 10^{-11}$ m at the microlenses diameter $D = 3$ mm, distance between their centers $d = 21$ mm, path length $L = 300$ m, and outer scale $L_0 = 10$ mm. In case of cross correlation,

$$\sigma_d^2 = 1.54207 \cdot 10^{-11}\ \text{m}^2,\ C_n^2 = 5.76473 \cdot 10^{-14}\ \text{m}^{-2/3}.$$

In numerical experiments, $h = 12$. For the period $\Delta t = 100$ s, $D = 3$ mm, $L = 300$ m, $f \approx 2$ m, and $\theta = 45°$, the path-averaged cross component of the wind velocity

$$V = \frac{0.213\sqrt{2}}{100\,\text{s} \cdot 0.855}\left(1.46(3.0 \cdot 10^{-3}\text{m})^{-1/3} - \right.$$

$$\left. - \frac{1.54207 \cdot 10^{-11}\text{m}^2}{2 \cdot 4.0\,\text{m}^2 \cdot 300\,\text{m} \cdot 6.48954 \cdot 10^{-14}\ \text{m}^{-2/3}}\right)^{-3} = 0.965\,\text{m/s}.$$

This value answers a wind velocity value of 1.0 m/s in the numerical dynamic turbulence model, which is used in the “frozen” turbulence movement modeling.

References

1. R. Hosfeld, J. Opt. Soc. Am. **44**, 284 (1954).
2. D. Fried, J. Opt. Soc. Am. **55**, No. 11, 1426–1435 (1965).
3. M. Sarazin and F. Roddier, Astronom and Astrophys. **227**, No. 1, 294–300 (1990).
4. A. Ziad, R. Conan, A. Tokovinin, F. Martin, and J. Borgnino, Appl. Opt. **39**, No. 10, 5415–5425 (2000).
5. A. Tokovinin, Appl. Opt. **41**, No. 10, 957–964 (2002).
6. L.V. Antoshkin, N.N. Botygina, O.N. Emaleev, L.N. Lavrinova, and V.P. Lukin, Atmos. Oceanic Opt. **11**, No. 11, 1046–1050 (1998).
7. A.S. Gurvich, A.I. Kon, V.L. Mironov, and S.S. Khmelevtsov, *Laser Radiation in Turbulent Atmosphere* (Nauka, Moscow, 1976), 277 pp.
8. V.P. Lukin and B.V. Fortes, *Adaptive Formation of Beams and Images in the Atmosphere* (Publishing House of SB RAS, Novosibirsk, 1999), 211 pp.
9. L.N. Lavrinova, Proc. SPIE **6160**, 201–205 (2005).