Light scattering by a finite-length cylinder in Wentzel–Kramers–Brillouin approximation. 1. Light scattering amplitude

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Approximate expressions for calculation of light scattering amplitude by optically "soft" circular cylinder of finite length have been derived in Wentzel–Kramers–Brillouin approximation for the case of light incidence perpendicularly to the cylinder axis. Scattering phase functions (or f_{11} element of the scattering phase matrix) of an infinitely long circular cylinder (rigorous solution) and of a finite-length circular cylinder calculated in Wentzel–Kramers–Brillouin approximation have been compared numerically.

Investigation of light scattering by natural and artificial aerosols, ice crystals, suspensions of biological particles is yet of a significant interest, in spite of a great number of experimental and theoretical studies performed.[1,2] Particles of such media often are non-spherical, so detailed study of theoretical characteristics of light scattering by such particles is quite important.

For describing light scattering by optically "soft" ($|m-1| \ll 1$, where m is the relative refractive index of the particulate matter) particles of non-spherical shape it is convenient to use the approach by Rayleigh–Gans–Debye (RGD), anomalous diffraction (AD)[3] and Wentzel–Kramers–Brillouin (WKB) approximations. The case of incidence of light along the symmetry axis of a circular cylinder of finite length in WKB approximation has already been considered earlier.[4]

Analysis of the amplitude and scattering phase function of a circular cylinder of finite length in the Wentzel–Kramers–Brillouin approximation for a light beam incident perpendicularly to the cylinder symmetry axis is presented in this paper.

1. Light scattering amplitude

Let us assume that the symmetry axis of an immobile homogeneous cylinder of the height H and radius a is oriented along the z-axis. Let plane electromagnetic wave falls along y-axis in Cartesian rectangular coordinates.

Let us use the integral representation of the light scattering amplitude in WKB approximation[5]:

$$\mathbf{f}(\mathbf{s},\mathbf{i}) = \frac{k^2}{4\pi}\left[-\mathbf{s}\times(\mathbf{s}\times\mathbf{e}_i)\right]\times$$

$$\times\int(m^2-1)T\exp\left[ik\left((m-1)(y'-y_1)+\mathbf{r}'(\mathbf{i}-\mathbf{s})\right)\right]\mathrm{d}V', \quad (1)$$

where $\mathbf{s}$ and $\mathbf{i}$ are the unit vectors along the direction of light scattering and propagation, respectively, $y_1 = -\sqrt{a^2 - x'^2}$ is the input coordinate of the particle surface for the wave passing through the point $\mathbf{r}'$, $k = 2\pi/\lambda$ is the wave number, λ is the wavelength in the disperse medium, $\mathbf{e}_i$ is the unit vector along the direction of polarization of the incident wave, $T = 2/(m+1)$ is the transmission coefficient; $\mathbf{r}'$ is the radius vector of a point inside the particle.

After integrating Eq. (1) over z in a cylindrical system of coordinates in a scalar form we have

$$f(\beta) = \frac{kaH\Delta}{4\pi}\times$$

$$\times\int_0^1\int_0^{2\pi}\exp\left[i\left(ka\psi_1(\beta,t,\varphi)+\frac{\Delta}{2}\psi_2(t,\varphi)\right)\right]\mathrm{d}\varphi\, t\,\mathrm{d}t, \quad (2)$$

where

$$\psi_1(\beta,t,\varphi) = t(\sin\beta\cos\varphi + [m-\cos\beta]\sin\varphi),$$

$$\psi_2(t,\varphi) = \sqrt{1-t^2\cos^2\varphi},$$

and $\Delta = 2ka(m-1)$ is the phase shift, β is the scattering angle read out from the forward scattering direction.

At small phase shifts $\Delta \ll 1$ one can completely ignore the value $\psi_2(t,\varphi)$ in comparison with $\psi_1(\beta,t,\varphi)$ in Eq. (2). Integration of Eq. (2) yields

$$f^{(0)}(\beta) = (ka)^2 H(m-1)\frac{J_1\left[kap(\beta)\right]}{kap(\beta)}. \quad (3)$$

Here $J_1(x)$ is the Bessel function of the first order, $p(\beta) = \sqrt{\sin^2\beta + (m-\cos\beta)^2}$.

In the RGD approximation[6]:

$$f^{\mathrm{RGD}}(\beta) = \frac{(ka)^2 H(m^2-1)}{2}\frac{J_1\left[kad(\beta)\right]}{kad(\beta)}, \quad (4)$$

where $d(\beta) = \sqrt{\sin^2\beta + (1-\cos\beta)^2} = 2\sin(\beta/2)$.

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As $m \to 1$ and $m^2 - 1 \approx 2(m-1)$ for optically "soft" particles, the formulas for the light scattering amplitude in the WKB approximation at small phase shifts (3) and in the RGD approximation (4) practically coincide.

Other approximations for calculation of the amplitude of light scattering in WKB approximation can be derived from Eq. (2), by expanding the phase $\psi_2(t,\varphi) = \sqrt{1 - t^2\cos^2\varphi}$, into a series, i.e.,

$$\psi_2(t,\varphi) = 1 - \frac{(t\cos\varphi)^2}{2} - \frac{(t\cos\varphi)^4}{8} - \frac{(t\cos\varphi)^6}{16} + O\left((t\cos\varphi)^8\right). \quad (5)$$

In particular, if $\psi_2(t,\varphi) = 1 - t^2/4$, we have the first approximation. Finally, using the expansion from Ref. 7 we obtain:

$$f^{(1)}(\beta) = \frac{kaH\exp\left[i\frac{\Delta}{2}\right]}{2\pi} \times \left[U_2\left(\frac{\Delta}{4}, kap(\beta)\right)h_0^{(2)}\left(\frac{\Delta}{8}\right) - U_1\left(\frac{\Delta}{4}, kap(\beta)\right)h_0^{(1)}\left(\frac{\Delta}{8}\right)\right], \quad (6)$$

where

$$U_1(x,y) = \sum_{s=0}^{\infty}(-1)^s\left(\frac{x}{y}\right)^{2s+1}J_{2s+1}(y),$$

$$U_2(x,y) = \sum_{s=0}^{\infty}(-1)^s\left(\frac{x}{y}\right)^{2s+2}J_{2s+2}(y)$$

are Lommel functions;

$$h_0^{(1)}(x) = j_0(x) + i\,y_0(x), \quad h_0^{(2)}(x) = j_0(x) - i\,y_0(x)$$

are the spherical Hankel functions of the 1st and 2nd kind.

If $\psi_2(t,\varphi) = 1 - \dfrac{t^2}{4} - \dfrac{t^2\cos2\varphi}{4}$, we have the second approximation.

Using Jacobi identity and after transformations we obtain from Eq. (2) that

$$f^{(2)}(\beta) = \frac{kaH\exp\left[i\frac{\Delta}{2}\right]}{2\pi}\left[R_0(c,b) + S_1(c,b) + S_2(c,b)\right]. \quad (7)$$

Here

$$S_1(c,b) = 2\sum_{j=1}^{\infty}(-1)^j\cos(4j\gamma)R_{2j}(c,b),$$

$$S_2(c,b) = 2i\sum_{j=0}^{\infty}(-1)^j\cos\left((4j+2)\gamma\right)R_{2j+1}(c,b),$$

$$R_n(c,b) = \int_0^1 J_n(cx)J_{2n}(b\sqrt{x})e^{-icx}\mathrm{d}x,$$

and

$$b = kap(\beta); \quad c = \frac{\Delta}{8}; \quad \gamma = \arctan\left(\frac{m - \cos\beta}{\sin\beta}\right).$$

Let us also present the integral $R_n(c,b)$ in the form of the series:

$$R_n(c,b) = \left(\frac{b^2c}{8}\right)^n \frac{1}{n!}\sum_{j=0}^{\infty}\frac{(-1)^j\left(b^2/4\right)^j}{j!(2n+j)!} \times \sum_{k=0}^{\infty}\frac{\Gamma\left(n+k+\frac{1}{2}\right)\Gamma(2n+1)(-2ic)^k}{\Gamma\left(n+\frac{1}{2}\right)\Gamma(2n+1+k)(2n+1+k+j)k!}, \quad (8)$$

where $\Gamma(z)$ is the Euler gamma function.

Numerical estimation of the error in calculation of the light scattering amplitude using WKB approach was carried out using the approximate formulas $f^{(1)}(\beta)$ and $f^{(2)}(\beta)$ by comparing those with the corresponding functions calculated directly by Eq. (2) at different phase shifts Δ and at different number of terms of the series in the Lommel functions $S_1(c, b)$ and $S_2(c, b)$. The results of comparison for the relative refractive index $m = 1.1$ are shown in Tables 1 and 2. The relative error here and below was calculated as $(F_{\text{approx}}/F_{\text{accurate}} - 1)\cdot 100\%$.

Table 1. Relative error in the light scattering amplitude $f^{(1)}(\beta)$ calculated using WKB approximate formula (6) in comparison with that calculated by Eq. (2) for a cylinder at different phase shifts Δ and scattering angles β

Δ	Scattering angle β, deg.					
	0		45		90	
	Re	Im	Re	Im	Re	Im
1	−0.36	3.52	−15.29	−131.60	−0.13	−0.78
2	−3.09	4.66	−18.89	62.35	−0.91	1.60
3	−22.92	5.71	−38.68	29.44	−6.57	7.05
5	105.85	−10.95	−136.28	5.03	13.05	0.41
6	147.69	−73.55	−689.99	−17.38	54.36	26.65

Table 2. Relative error in the light scattering amplitude $f^{(2)}(\beta)$ calculated using WKB approximate formula (7) in comparison with that calculated by Eq. (2) for a cylinder at different phase shifts Δ and scattering angles β

Δ	Scattering angle β, deg.					
	0		45		90	
	Re	Im	Re	Im	Re	Im
1	−0.43	3.10	−5.03	−53.63	−0.84	7.24
2	−2.37	3.14	−8.10	33.69	−3.46	3.74
3	−11.51	3.18	−17.40	18.04	−14.78	4.94
5	32.50	2.25	−68.18	11.17	31.68	−2.59
6	46.86	−1.49	−368.27	1.99	36.54	−1.62

The light scattering amplitude $f^{(1)}(\beta)$, in the first approximation, provides quite reliable results with the error less than 10% (both for real and imaginary parts) only at small phase shifts $\Delta < 3$ for the forward scattering (see Table 1). The second approximation of the light scattering amplitude $f^{(2)}(\beta)$ is also realistic at large phase shifts $\Delta < 10$ (see Table 2). It is revealed that the approximate formulas for amplitudes $f^{(1)}(\beta)$ and $f^{(2)}(\beta)$ calculated by WKB approximation at $\Delta < 3$ have good convergence, in particular, one or two terms of the series are sufficient in the Lommel functions $S_1(c, b)$ and $S_2(c, b)$ with the error less than 1%.

2. Light scattering phase function

The light scattering phase function (or the element f_{11} of the scattering phase matrix) for natural light (polarization is random) was calculated by the following formula

$$f_{11}(\beta) = \frac{1+\cos^2(\beta)}{2} k^2 |f(\beta)|^2, \qquad (9)$$

where $|f(\beta)|^2$ is the square of the absolute value of the light scattering amplitude.

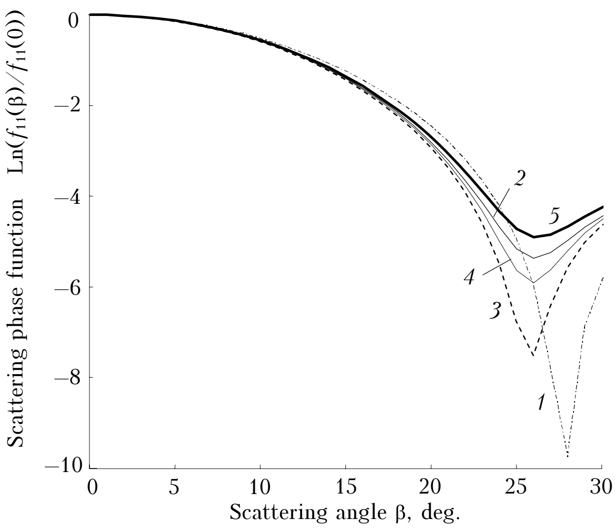

Fig. 1. Scattering phase function $\mathrm{Ln}[f_{11}(\beta)/f_{11}(0)]$ as a function of the scattering angle β for a finite cylinder calculated using the light scattering amplitudes: RGD (curve *1*), WKB $f(\beta)$ (*2*), WKB $f^{(1)}(\beta)$ (*3*), WKB $f^{(2)}(\beta)$ (*4*), and for an infinitely long cylinder (*5*) with the relative refractive index $m = 1.1$ at $ka = 8$.

Calculations for an infinitely long cylinder (exact solution) were carried out by the algorithm from Ref. 8, and in the WKB approximation, i.e., using the light scattering amplitudes (2), (6), and (7). The scattering phase function was normalized to that for the forward scattering direction.

Numerical comparison has shown that the value of the relative error in the scattering phase function calculated using WKB approximation, as compared with the exact solution for an infinitely long cylinder, does not exceed 9% in the range of scattering angles up to 25°. The scattering phase functions calculated using RGD and WKB approximations and the exact one calculated for an infinitely long cylinder with the relative refractive index $m = 1.1$ and $ka = 8$ are shown in Fig. 1. Obviously, the approximate amplitude $f^{(2)}(\beta)$ provides better approach of the WKB scattering phase function (see figure) to that by the exact solution, than the approximate amplitude $f^{(1)}(\beta)$.

Conclusion

The approximate formulas have been derived for calculation of light scattering amplitude calculated by WKB approximation for an optically "soft" cylinder of finite length, when the light is incident perpendicularly to the cylinder symmetry axis. The numerical results on the light scattering phase function of a circular cylinder in WKB approximation is compared with the exact solution for an infinitely long cylinder. The relative error of the scattering phase function for cylinder in WKB approximation, as compared with that by the exact solution for infinitely long cylinder does not exceed 9% in the range of scattering angles up to 25°.

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