Light scattering by a finite-length cylinder in Wentzel–Kramers–Brillouin approximation. 2. Efficiency factors of extinction and absorption

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The formulas for calculation of the efficiency factors of extinction and absorption by optically "soft" ($|m-1| \ll 1$, where $m = n - i\chi$ is a relative refractive index of the cylinder) circular cylinder of a finite length for incident light direction perpendicular to the axis of cylinder in Wentzel–Kramers–Brillouin (WKB) approximation are obtained. It is shown analytically and numerically that the values of factor efficiency extinction by circular cylinder in WKB and anomalous diffraction (AD) approximations fully coincide. It is noted that unlike extinction efficiency factor only under significant optical "softness" of the circular cylinder ($|m-1| \to 0$) the expression for efficiency factor of absorption by circular cylinder in WKB approximation passes into such expression in AD approximation.

The study of light scattering by non-spherical particles entering into composition of natural and artificial aerosols, suspensions of biological particles, and ice crystals has a great practical significance for monitoring the atmosphere and ocean state; for colloid chemistry, and so on.[1,2]

It is convenient to use for optically "soft" ($|m-1| \ll 1$, where $m = n - i\chi$ is the relative refractive index of a particle) light scattering particles of non-spherical shape the Rayleigh–Gans–Debye (RGD), anomalous diffraction (AD) [Ref. 3] and Wentzel–Kramers–Brillouin (WKB) approximations.

This paper presents the analysis of the extinction and absorption efficiency factors of a finite round cylinder in the Wentzel–Kramers–Brillouin approximation for the light beam incident perpendicularly to the cylinder symmetry axis.

1. Extinction efficiency factor

Using the integral representation of the light scattering amplitude in the WKB approximation[4,5] in the scalar form we have:

$$f(\beta) = \frac{\rho^2 H(m-1)}{2\pi} \times$$
$$\times \int_0^1 \int_0^{2\pi} \exp\left[i\left(\rho\psi_1(\beta,t,\varphi) + \frac{\Delta}{2}\psi_2(t,\varphi) \right) \right] \mathrm{d}\varphi t \mathrm{d}t, \quad (1)$$

where

$$\psi_1(\beta, t, \varphi) = t(\sin\beta\cos\varphi + [m - \cos\beta]\sin\varphi),$$
$$\psi_2(t,\varphi) = \sqrt{1 - t^2\cos^2\varphi};$$

a is the cylinder radius; H is the cylinder height; $\Delta = 2\rho(m-1)$ is the phase shift; $\rho = ka$ is the diffraction parameter of the cylinder; $k = 2\pi/\lambda$ is the wave number; λ is the wavelength in the dispersion medium; β is the scattering angle counted from the forward scattering direction.

Consider the case of non-absorbing cylinders ($\chi = 0$).

For the forward scattering direction ($\beta = 0°$), at small phase shifts $\Delta < 1$, one can expand Eq. (1) into Taylor series in terms of Δ. After integration we have

$$f(0) = \frac{\rho H}{2\pi}\left[\pi\Delta\left(\frac{1}{2} - \frac{\Delta^2}{16} + \frac{\Delta^4}{384} + O(\Delta^6) \right) + \right.$$
$$\left. + i\Delta^2\left(\frac{2}{3} - \frac{2\Delta^2}{45} + \frac{2\Delta^4}{1575} + O(\Delta^6) \right) \right]. \quad (2)$$

For any phase shifts

$$f(0) = \frac{\rho H}{2}\left[J_1(\Delta) + iH_1(\Delta) \right], \quad (3)$$

where $J_1(x)$ is the Bessel function of the first type; $H_1(x)$ is the Struve function.

According to the optical theoreme[5], one can write the extinction cross section σ_e normalized to the area S of the projection of particle on the plane, perpendicular to the beam axis (or the extinction efficiency factor Q_e) in the form

$$Q_e = \frac{\sigma_e}{S} = \frac{4\pi}{k\,S}\mathrm{Im}\left(\mathbf{f}(\mathbf{i},\mathbf{i})\right)\cdot\mathbf{e}_i, \quad (4)$$

where $\mathbf{i}$, $\mathbf{e}_i$ are the unit vectors in the direction of propagation and polarization of the incident wave, respectively.

Using the area $S = 2aH$, the light scattering amplitude [Eq. (3)], we have for non-absorbing cylinders

$$Q_e = \pi H_1(\Delta). \quad (5)$$

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Note that Eq. (5) coincides with that in AD approximation.[6] The dependences of the extinction efficiency factor Q_e on the phase shift Δ calculated by Eq. (5) and by means of approximate formulas for WKB amplitudes $f^{(1)}(\beta)$ and $f^{(2)}(\beta)$ [Ref. 4], as well as for an infinitely long cylinder (rigorous solution)[7] with relative refractive index $m = 1.1$ are shown in the Fig. 1.

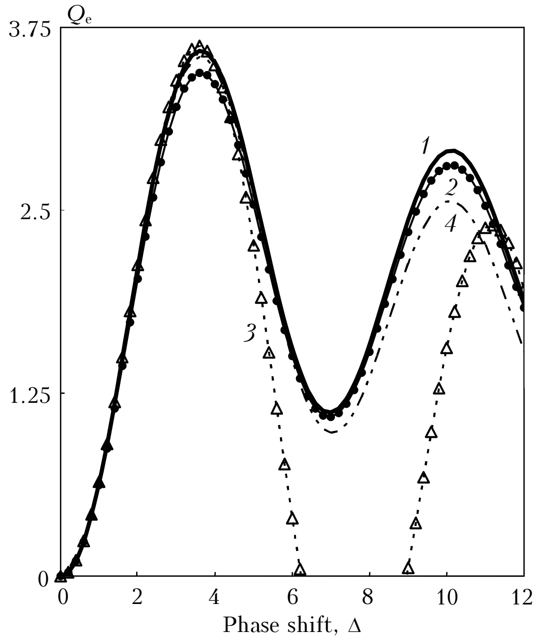

Fig. 1. Dependence of the extinction efficiency factor Q_e on the phase shift Δ for infinitely long non-absorbing cylinder with relative refractive index $m = 1.1$ (curve *1*) and using the WKB scattering amplitudes in the optical theorem: $f(\beta)$ (curve *2*), $f^{(1)}(\beta)$ (curve *3*), and $f^{(2)}(\beta)$ (curve *4*).

It is seen that the values of the extinction efficiency factors Q_e calculated using the approximate formulas for WKB amplitudes $f^{(1)}(\beta)$ and $f^{(2)}(\beta)$, starting from the phase shift of 4, are underestimated relative to the extinction efficiency factors for infinitely long cylinder (see the figure).

In the case of absorbing cylinders ($\chi > 0$), the formula for the extinction efficiency factor Q_e in WKB approximation has more complicated form than Eq. (5), although it is analytically reduced to the AD approximation.[8] Numerical examination of calculation of the extinction efficiency factor Q_e in WKB approximation by Eqs. (1) and (4) for the phase shifts $0 < \Delta < 100$ and $0 < \chi < 1$ has confirmed its complete coincidence with AD approximation[3]:

$$Q_e = 2\left[1 - \int_0^1 \cos\left(2\rho(n-1)\sqrt{1-t^2}\right) \times \exp\left(-4\rho\chi\sqrt{1-t^2}\right)\mathrm{d}t\right]. \quad (6)$$

At small phase shifts $\Delta < 1$ and $\rho\chi < 1$, we obtain from Eq. (6):

$$Q_e = 2\left[1 - \frac{\sqrt{\pi}}{2}\sum_{k=0}^{\infty}\frac{(-1)^k\Delta^{2k}}{(2k)!}\left(\sum_{j=0}^{\infty}\frac{(-1)^j(2\rho\chi)^j\,\Gamma\left(\frac{2k+2+j}{2}\right)}{j!\,\Gamma\left(\frac{2k+1+j}{2}+1\right)}\right)\right], \quad (7)$$

where $\Gamma(x)$ is the Euler gamma function.

2. Absorption efficiency factor

According to Ref. 5, the absorption cross section σ_a normalized to the area S of the projection of particle on the plane perpendicular to the beam axis (or the absorption efficiency factor Q_a) is equal to:

$$\frac{\sigma_a}{S} = Q_a = \frac{\int_V k\varepsilon_r''|\mathbf{E}(\mathbf{r})|^2\,\mathrm{d}V}{S}, \quad (8)$$

where $\mathbf{E}(\mathbf{r})$ is the total electric field inside the particle; $\varepsilon_r'' = 2n\chi$ is the imaginary part of the relative permittivity of the particle.

If the incident wave has the form $\mathbf{E}_i(\mathbf{r}) = \mathbf{e}_i\exp[iky]$, one can write the total electric field inside a cylinder in the WKB approximation in the form (see Refs. 4 and 5):

$$\mathbf{E}(\mathbf{r}) = \frac{2\mathbf{e}_i\exp\left\{ik\left[m\left(y+\sqrt{a^2-x^2}\right)-\sqrt{a^2-x^2}\right]\right\}}{m+1}. \quad (9)$$

Thus, using Eqs. (8) and (9), we obtain the formulas for the absorption efficiency factor in WKB approximation:

$$Q_a^{\mathrm{WKB}} = \frac{4n}{(n+1)^2+\chi^2}\left[\frac{\pi}{2}\left(I_1(4\rho\chi) - L_1(4\rho\chi)\right)\right], \quad (10)$$

where $I_1(x) = -iJ_1(ix)$ is the modified Bessel function of the first type; $L_1(x) = -H_1(ix)$ is the modified Struve function.

At small $\rho\chi \ll 1$ we obtain from Eq. (10), expanding in the series:

$$Q_a^{\mathrm{WKB}} = \frac{4n}{(n+1)^2+\chi^2} \times \left[\pi\rho\chi - \frac{16}{3}(\rho\chi)^2 + 2\pi(\rho\chi)^3 - \frac{256}{45}(\rho\chi)^4 + O\left((\rho\chi)^5\right)\right]. \quad (11)$$

Note that the formula for the absorption efficiency factor in AD approximation[3,6] has the form:

$$Q_a^{\mathrm{AD}} = 1 - \int_0^1 \exp\left(-4\rho\chi\sqrt{1-t^2}\right)\mathrm{d}t = \frac{\pi}{2}\left(I_1(4\rho\chi) - L_1(4\rho\chi)\right). \quad (12)$$

For optically "soft" cylinders $4n/[(n+1)^2+\chi^2] \to 1$, i.e., the formula for the absorption efficiency factor (10) in the WKB approximation is transformed to that in the AD approximation [Eq. (12)]. At $n = 1.3$ and $\chi = 0.1$ the values of the absorption efficiency factor [Eq. (10)] in the WKB approximation are underestimated relative to the AD approximation [Eq. (12)] by 2%; at $n = 1.1$ and $\chi = 0.1$ the underestimation is already 0.5%.

Conclusion

The formulas are obtained for calculation of the extinction and absorption efficiency factors of optically "soft" round cylinder of finite length at the light incidence perpendicularly to the cylinder symmetry axis in the WKB approximation. The formulas for extinction and absorption efficiency factors of the round cylinder in the WKB and the AD approximation are analyzed.

Some insignificant differences are noted in the extinction efficiency factors of round cylinder in WKB and AD approximations. The formulas for the absorption efficiency factors in the WKB approximation obtained according to the optical theorem and in the AD approximation completely coincide.

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