MATHEMATICAL MODELS OF BROKEN CLOUDINESS TAKING INTO ACCOUNT THE RANDOM GEOMETRY OF THE INDIVIDUAL CLOUDS

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New mathematical models of stochastically nonuniform cloud fields are considered taking into account a random geometry of individual clouds. These models are constructed on the basis of sum of independent random Gaussian fields with decreasing variances and correlation lengths that is essentially close to the so–called cascade processes, which are used in simulations of the fractal clouds. The cloud fields which are simulated on the basis of Poisson and sum of normal cloud fields are most close to actual cloud fields. Within the context of the method of numerical simulation of cloud and radiation fields the algorithms of the Monte Carlo method for calculation of linear functionals of mean intensity have been developed and the effect of the random geometry of individual clouds on the mean fluxes of visible solar radiation have been evaluated.

At present the need for taking into account the stochastically geometrical structure of cumulus cloudiness field have been generally recognized when calculating the radiation fluxes and the brightness fields of the atmosphere–underlying surface system. In existing mathematical models of cumulus clouds individual clouds are usually approximated by the simplest geometric bodies (such as cylinders, parallelepipeds, truncated paraboloids of rotation, and the like), whereas the shape of real cumulus clouds is highly irregular and varies over a wide range of scales. The geometric objects of such a structure are commonly referred to as fractals, while for describing them the mathematical apparatus of the theory of measure of sets with nonintegral (fractal) Hausdorff dimensionality[1,2] is employed. Description of some methods for simulating the fractal surfaces can be found in Ref. 3.

In Refs. 4 and 5 the n–step cascade processes are proposed for simulating a cloud field. It is obvious advantage of the cascades that they make it possible to construct the clouds of a prescribed fractal dimension which for the real cumulus clouds is inferred from satellite data.[6–9] However, simulation of cascade processes is highly laborious and demands very large amount of computing resources for numerical construction of cloud fields even in relatively small spatial regions, that makes it practically impossible to calculate the statistical characteristics of radiation field by averaging the solution of stochastic transfer equation over an ensemble of such fields.

In this paper some simpler mathematical models of cumulus clouds, which are close to the cascade ones and take into account the random geometry of individual clouds, are proposed to be constructed on the basis of a sum of the uniform, isotropic Gaussian fields with prescribed correlation functions. Within the scope of the method of numerical simulation of cloud and radiation fields the effect of random geometry of individual cumulus clouds on the mean fluxes of visible solar radiation is evaluated.

GAUSSIAN MODEL OF CUMULUS CLOUDINESS

For better understanding we present a brief description of a Gaussian model of cumulus cloudiness,[9] based on which we will construct the mathematical models taking into account an irregular random geometry of individual clouds.

In this model we assumed that the lower boundary of clouds is in the plane $z = h_0$, while the upper boundary is given by the relation

$$\zeta(x, y) = h_0 + \max(|\upsilon(x, y)| - c,\ 0)\ ,\quad c > 0\ , \tag{1}$$

where $\upsilon(x, y)$ is the uniform Gaussian field with zero mean, correlation function $K(x, y)$, and variance $\sigma^2 = K(0, 0)$. The input parameters of the model (c, σ, and correlation function $K(x, y)$) may be related to the experimentally determined quantities, such as cloud amount index N and the mean horizontal D and vertical H dimensions of clouds.

It is not difficult to show that the cloud amount index is determined from the relation

$$N = 2(1 - \Phi(c/\sigma))\ , \tag{2}$$

where $\Phi(x)$ is the function of standard normal distribution. Mean diameter can be determined from the relation derived based on formulas (45)–(51) from Ref. 10

$$D = \sigma\,\frac{4\sqrt{2\pi}\ N(c/\sigma)^{-1}\ \exp(c^2/2\sigma^2)}{\sqrt{k_{20}k_{02} - k_{11}^2}}\ , \tag{3}$$

where

$$k_{ij} = -\left.\frac{\partial K(x,\ y)}{\partial x^i \partial y^j}\right|_{x=y=0}\ .$$

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Let us assume we know that the function of mean height of clouds $\hat{H}(d)$ depending on the parameter d from an auxiliary model

$$\hat{\omega}(x, y) = h_0 + \max(|\hat{\upsilon}(x, y)| - d, 0) ,$$

where $\hat{\upsilon}(x, y)$ is the uniform Gaussian field having the same normalized correlation function, as $\upsilon(x, y)$ but with the unit variance.

Then for the mean cloud height the following equality is obvious:

$$H(c) = \sigma\hat{H}(c/\sigma). \tag{4}$$

In what follows the cloud fields are assumed to be isotropic. This is rational for two reasons: first, nonisotropic structures can be easily constructed from isotropic analogs by changing the scale along one of the coordinate axes, and, second, the algorithm of calculations is simplified. In this case in Ref. 11 for $\hat{H}$ the relation

$$\hat{H}(d) = \int\limits_d^\infty (h - d)\, p(h)\, \mathrm{d}h \Big/ \int\limits_d^\infty p(h)\, \mathrm{d}h , \tag{5}$$

has been employed in which the probability density of the magnitudes of local maxima in accordance with Ref. 12 has the from

$$p(h) = \frac{\sqrt{3}}{2\pi}\left\{\exp\left(\frac{-\alpha h^2}{2\alpha - 3}\right) h\sqrt{\frac{3(2\alpha - 3)}{\alpha^2}} + \right.$$

$$+ \frac{3\sqrt{2\pi}}{2\alpha} \exp(-h^2/2)(h^2 - 1)(1 + \mathrm{erf}(\beta)) + \sqrt{2\pi}\, \times$$

$$\left. \times\sqrt{\frac{\alpha}{3(\alpha - 1)}} \exp\left(\frac{-\alpha h^2}{2(\alpha - 1)}\right)(1 + \mathrm{erf}(\gamma))\right\} , \tag{6}$$

$$\beta = h\sqrt{\frac{3}{2(2\alpha - 3)}} , \quad \gamma = h\sqrt{\frac{\alpha}{(2\alpha - 2)(2\alpha - 3)}} ,$$

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int\limits_0^x \exp(-t^2)\, \mathrm{d}t , \quad \alpha = \sigma^2 k_{40}/k_{20}^2 .$$

The problem on numerical construction of the model is reduced to simulation of the uniform isotropic Gaussian field with a prescribed correlation function, for whose approximate simulation the method of randomization of spectrum[13] has been proposed in Ref. 9 in a modification for the case of isotropic fields. The spectral measure of isotropic field on a plane is circulary symmetrical, and the correlation function has the form

$$K(x, y) = K(r) = \sigma^2 \int\limits_0^\infty J_0(\rho r)\, \mu(d\rho) ,$$

where $r^2 = x^2 + y^2$, $J_0(z)$ is the Bessel function of the first kind with zeroth subscript, $\mu(d\rho)$ is the radially spectral measure on $[0, +\infty)$.

The spectral space is divided into sectors and concentric rings, the points of a randomized spectrum lie on circles, and the Gaussian field is approximately simulated according to the formula

$$\upsilon(x, y) = \frac{\sigma}{\sqrt{I}} \sum_{J=1}^{J} a_j \sum_{i=1}^{I} \sqrt{-\ln(\hat{\alpha}_{ij})} \cos((x\rho_j \cos\omega_i + y\rho_j \sin\omega_i) + 2\pi\hat{\beta}_{ij}), \tag{7}$$

where ρ_j and ω_i are the polar coordinates of the points of the spectrum, a_j are the coefficients associated with dividing the spectral space, and $\hat{\alpha}_{ij}$ and $\hat{\beta}_{ij}$ are independently random quantities uniformly distributed on [0, 1).

Let the spectral measure be concentrated on the circle with the radius ρ. Then we obtain for the correlation function $K(r) = \sigma^2 J_0(\rho r)$ and Eq. (7) takes the form

$$\upsilon(x, y) = \frac{\sigma}{\sqrt{I}} \sum_{i=1}^{I} \sqrt{-\ln(\hat{\alpha}_i)} \cos((x\rho\cos\omega_i + y\rho\sin\omega_i) + 2\pi\hat{\beta}_i), \tag{8}$$

For a given spectral measure $k_{20} = \sigma^2\rho^2/2$, $k_{40} = -3\sigma^2\rho^4/8$, $\alpha = 1.5$ and from Eq. (6) it follows

$$p(h) = \frac{2\sqrt{3}}{2\pi} (h^2 - 1 + \mathrm{e}^{-h^2})\, \mathrm{e}^{-h^2/2} , \quad h \geq 0 . \tag{9}$$

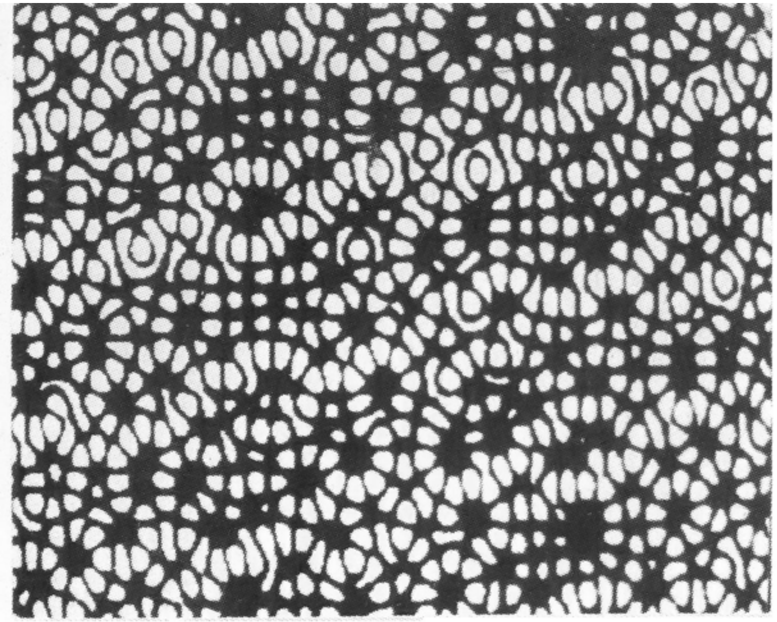
When prescribing the input parameters of the model, formulas (2)–(5) are used based on Eq. (9) and the corresponding values of k_{ij}, the value $\hat{H}(d)$ is estimated numerically.

For convenience the above–considered model of a cloud field will below referred to as the G_1 model. The horizontal and vertical cross sections of individual cloud fields constructed on the basis of Gaussian and Poisson[14] models are shown in Figs. 1 and 2. It can be seen from the figures that the vertical cross sections of clouds in the G_1 model are well approximated by the vertical cross–sections of truncated paraboloids. In contrast to the Poisson model, in the G_1 model the spatial structure of a cloud field looks less realistic because it is more regular (see Figs. 1*a* and 2*a*).

Irregular and odd shapes of real clouds are a product of chaotic motions of different scales which occur in the atmosphere. The above–considered model provides only more or less correct account of the contribution of rather large–scale atmospheric motions to the formation of a cloud geometry. For this reason the model describes only the general profiles of individual clouds and is unable to reproduce details of a fine geometric structure. This deficiency can be eliminated, if we sum up n independent Gaussian fields with decreasing variances and correlation lengths (G_n model) that makes it possible to take, to a certain degree, into account the effect of atmospheric motions of different scales on the geometric shapes of the modeled clouds.

In model (1) let $\upsilon(x, y) = \sum\limits_{i=1}^{n} \upsilon_i(x, y)$. Here $\upsilon_i(x, y)$ are independent uniform isotropic Gaussian fields with zero means and correlation functions $K_i(r) = \sigma J_0(\rho_i r)$, where $r = (x^2 + y^2)^{1/2}$, while σ_i and ρ_i satisfy the inequalities

$$\begin{cases} \sigma_{i+1} < \sigma_i, \\ 1/\rho_{i+1} < 1/\rho_i, \end{cases} \qquad i = 1, \ldots, n - 1 .$$



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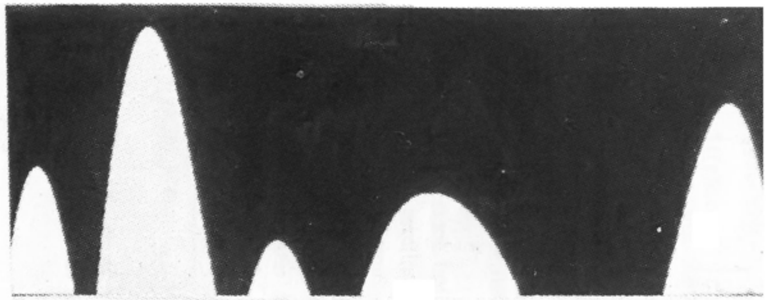

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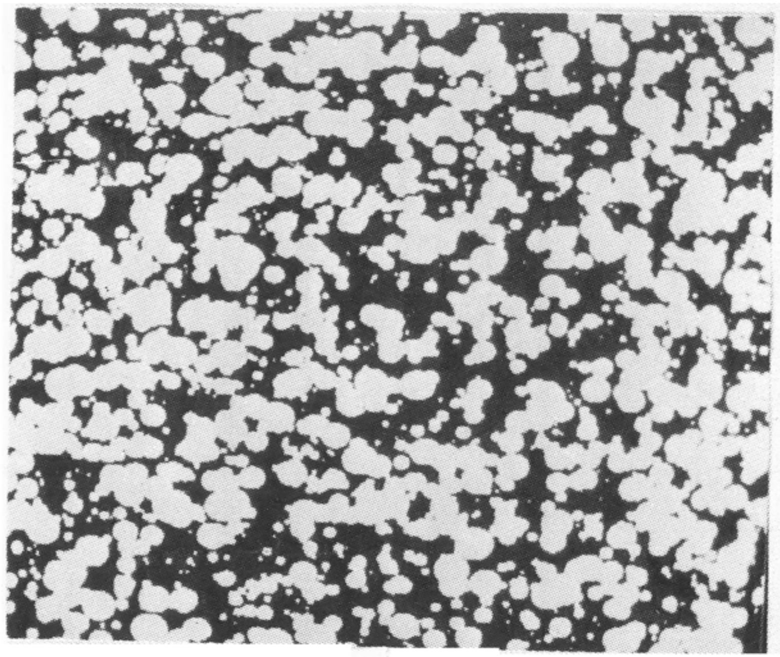
FIG. 1. A random cloud field constructed with the use of the G_1 model: a) horizontal cross section by the plane $z = h_0$ (region 25 *by* 25 km*) and b) vertical cross section (sampling length is* 3 km*).*



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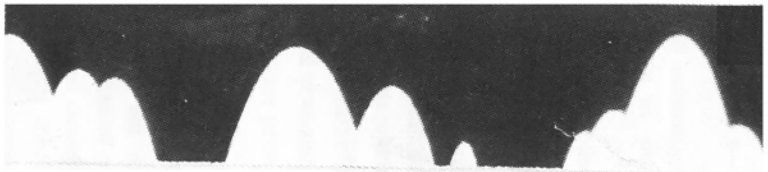

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FIG. 2. A random cloud field constructed using the Poisson model: a) horizontal cross section by the plane $z = h_0$ (region 25 *by* 25 km*) and b) vertical cross section (sampling length is* 5 km*).*

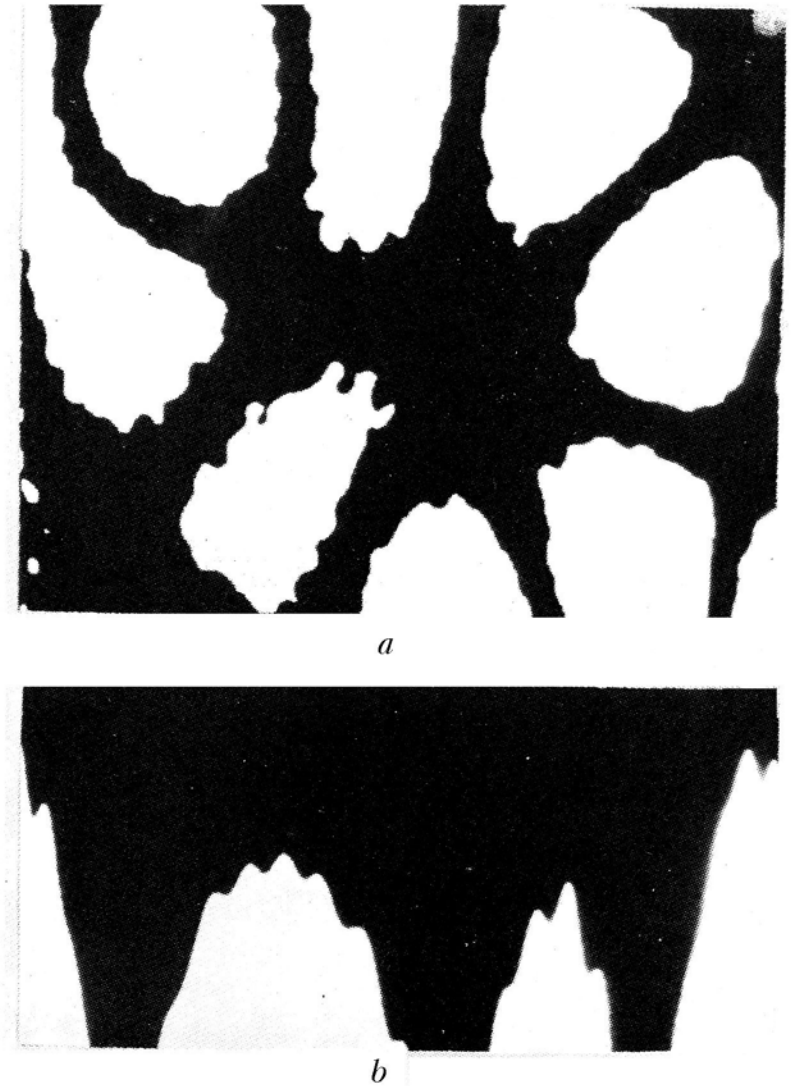

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FIG. 3. A random cloud field constructed with the use of the G_2 *model* ($\lambda = 0.005$ and $\mu = 0.1$), $N = 0.5$, *and* $D = H = 0.25$ km: *a) horizontal cross section by the plane* $z = h_0$ *(region* 25 *by* 25 km) *and b) vertical cross section (sampling length is* 1 km).

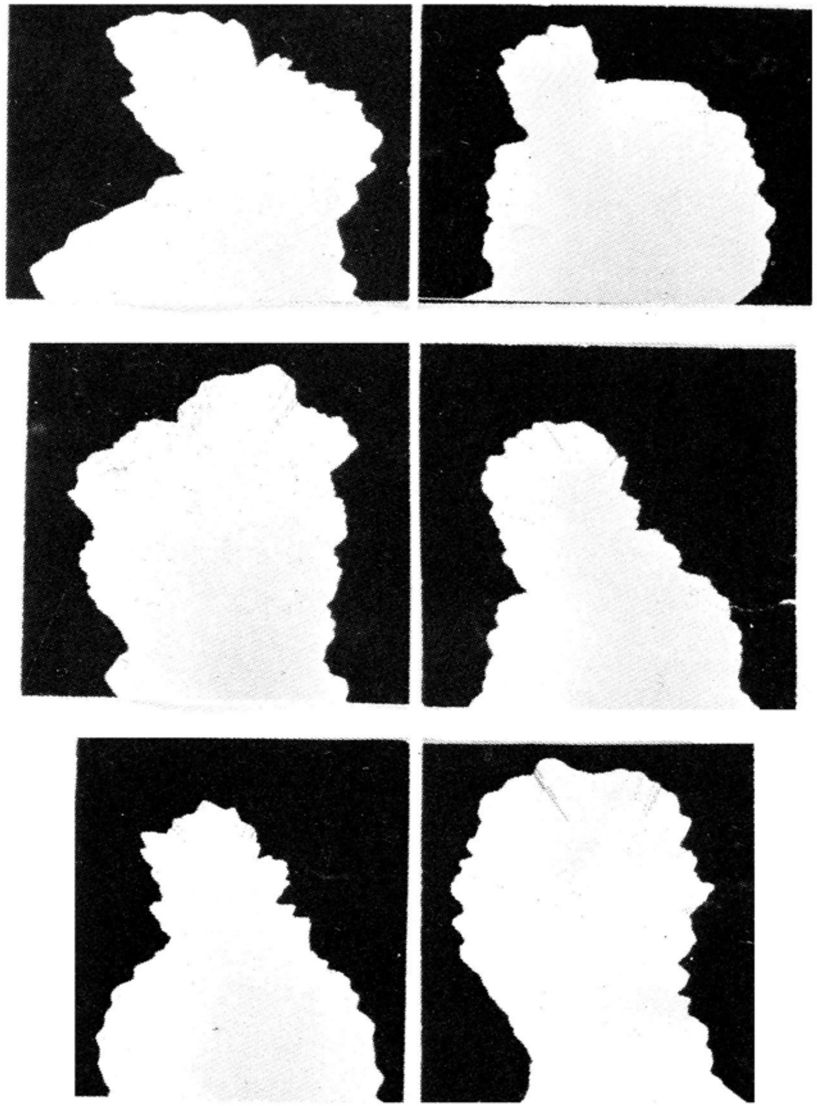

FIG. 4. Vertical cross sections of random images of individual clouds constructed using the PG_n *model.*

Since the summed fields are independent, for the variance and correlation function of the resulting field we have the relations

$$\sigma_\upsilon^2 = \sum_{i=1}^{n} \sigma_i^2 \ ; \ K_\upsilon(r) = \sum_{i=1}^{n} \sigma_i^2 \, J_0(\rho_i \, r) \ .$$

Note that because of the stability of the Gaussian distributions the field $\upsilon(x, y)$ is also Gaussian, and, therefore, relations (2)–(6) can be used for adjusting the model parameters in accordance with the known values of N, D, and H. However, the system is incomplete, and as result, only three model parameters can be determined unambiguously, such as σ_1, ρ_1, and c, while the remaining σ_2, ... σ_n and ρ_2, ... ρ_n must be assumed to be known *a priori*. For convenience, the latter can be replaced by λ_2, ... λ_n and μ_2, ... μ_n which are assigned *a priori* and relate to the variances and correlation lengths as follows:

$$\begin{cases} \sigma_i = \lambda_i \sigma_i, \\ 1/\rho_i = \mu_i/\rho_i, \end{cases} \qquad i = 2, \ldots, n \ . \qquad (10)$$

We will refer to the constructed model as a G_n model (where subscript n indicates the number of summed fields).

The algorithm of numerical simulation of cloud fields in the G_n model is evident, that is, n Gaussian fields are simulated independently according to formula (8), and then they are summed up. Images of cloudiness, constructed by summing up two Gaussian random fields are shown in Fig. 3.

In comparison with the G_1 model, in this case the individual clouds have more irregular shapes, which, generally speaking, still essentially differ from the geometric shapes of real clouds.

THE POISSON–GAUSSIAN MODEL OF CUMULUS CLOUDS

As was already mentioned, in comparison with the Poisson random fields, the Gaussian are more regular, but they enable one, in principle, to simulate the clouds of random geometric shapes. The efficiencies of both models can be united if one constructs the cloudiness based on the Poisson indicator field (P model) and the G_n model.

Let $\upsilon(x, y)$ be the sum of n independent Gaussian fields with decreasing correlation lengths and variances from the G_n model. Together with $\upsilon(x, y)$ we consider the Poisson indicator field to be consisting of a set of truncated paraboloids of rotation of the fixed diameter D_0 and the height H_0. The cloud centers $r_{01}, \ldots, r_{0m}$ which in our case coincide with the geometric centers of bodies, lie in the same plane $z = h_0$. Recall that a random number m of clouds is simulated in accordance with the Poisson distribution, while the coordinates of cloud centers are uniformly distributed over space.

Let $\mathbf{n}(r_{pj}) = (n_x(\mathbf{r}_{pj}), n_y(\mathbf{r}_{pj}),$ and $n_z(\mathbf{r}_{pj}))$ is the vector of outward normal at the point $\mathbf{r}_{pj} = (x_{pj}, y_{pj}, z_{pj})$ to the paraboloid with its center being at the point $\mathbf{r}_{0j}$, where $j = 1, \ldots, m$. Let us define a one–to–one representation L: $\mathbf{r}_{pj} = (x_{pj}, y_{pj}, z_{pj}) \to \mathbf{r}_j = (x_j, y_j, z_j)$ as follows:

$$\begin{cases} x_j = x_{pj} + n_x(\mathbf{r}_{pj}) \, |\upsilon(x_{pj}, y_{pj})| \\ y_j = y_{pj} + n_y(\mathbf{r}_{pj}) \, |\upsilon(x_{pj}, y_{pj})| \\ z_j = z_{pj} + n_z(\mathbf{r}_{pj}) \, |\upsilon(x_{pj}, y_{pj})| \end{cases} \quad j = 1, \ldots, m \ .$$

Such a representation, which in fact shows the "pulling" of the modulus of the sum of Gaussian fields over the paraboloids in the direction of the outward normal, transforms smooth paraboloids into some random geometric bodies. Bounding these bodies in their bottoms by the plane $z = h_0$, we obtain the mathematical model of cumulus cloudiness, in which individual clouds have a random geometry.

It is obvious that the resulting field is not Gaussian, and therefore the mathematical problem on the probabilistic properties of such a field remains unresolved. Due to the use of the nonlinear procedure L the solution of this problem is rather complicated. In particular, it is far from being a trivial task to derive the analytical expressions even for such relatively simple statistical characteristics as the mathematical expectation and correlation function. At the first stage of investigations it is useful, first of all, to evaluate the effect of stochastic geometry of individual clouds on the radiative characteristics of a cloud field. If this effect is strong then one should proceed to a more detailed physical interpretation and detailed investigation of probabilistic properties of the proposed mathematical model of broken clouds.

In this current model which subsequently will be referred to as the PG_n model, three groups of input parameters are used:

1) The parameters of the P model (without the sum of Gaussian fields "pulled", over the paraboloids), that is the diameter D_0 and the height H_0 of paraboloids, as well as the cloud amount index N_0 (which we will refer to as the term "initial" for convenience), giving the mean number ν of cloud centers by unit area.

2) The parameters σ_1 and ρ_1, which should be adjusted in accordance with the assigned values of D_0 and H_0.

3) The parameters λ_i, μ_i $i = 2, \ldots, n$ which are given simply as: $\lambda_i = 1/2^{i-1}$, $\mu_i = 1/2^{i-1}$.

Since the number of cloud centers is fixed while their mean horizontal dimensions increase when adding the modulus of a Gaussian field, the true cloud amount index $N \ge N_0$ and is calculated numerically. The mean cloud height is given by formula $\langle H\rangle = H_0 + H_g$, where $H_g = \sigma_\upsilon/\sqrt{2\pi}$ is the mean increase of cloud heights due to adding the Gaussian field $\upsilon(x, y)$. The quantity $\langle D^2\rangle$ is calculated by virtue of the following considerations. The mean number ν of cloud centers by unit area is given by formula[14]

$$n = \frac{4\ln\,(1 - N_0)}{\pi D_0^2} = \frac{4\ln\,(1 - N_0)}{\pi\langle D_0^2\rangle} \ .$$

Consequently,

$$\langle D_0^2\rangle = \frac{\ln\,(1 - N)\, D_0^2}{\ln\,(1 - N_0)} \ .$$

The problem of numerically constructing of the PG_n model is reduced to an independent simulation of two random fields. The algorithm of simulation of the uniform isotropic Gaussian field with a given correlation function was considered above. The algorithm of simulation of the Poisson indicator field can be found in Ref. 14. In Fig. 4 the images of clouds constructed using the PG_n model are shown. As can be seen from the figure, the obtained pictures are very interesting since these images are too close to the realistic cloud images.

CALCULATION OF THE RADIATIVE CHARACTERISTICS OF CUMULUS CLOUDINESS

The mean fluxes ($<A>$ is albedo, $<S>$ and $<Q_s>$ are the direct and transmitted scattered radiation, respectively) in the cumulus cloudiness were calculated for the radiation with the wavelength $\Lambda = 0.69$ μm. The scattering and absorption in the clear atmosphere were neglected. It was assumed that the clouds represent the scattering medium with $\Sigma = 30$ km^{-1}. The photon trajectories were simulated based on the method of maximum cross section. The calculations were performed for PG_1 model with the following input parameters: $N_0 = 0.1$, 0.3, and 0.5; $D_0 = 1.0$ km; $H_0 = 1.0$ km; $\sigma = 0.075$ km; and $\rho = 30$ km^{-1}. For given parameters of the model, the cloud amount index was taken to be $N = 0.13$, 0.373, and 0.593, and the mean cloud height $<H> = 1.06$ km and rms horizontal cloud size $<D^2>^{1/2} = 1.143$ km. In order to evaluate the effect of random geometry of cumulus clouds on the mean fluxes of visible solar radiation, the corresponding radiative characteristics for the P model were calculated simultaneously at the same cloud amount indicies N, horizontal size $<D^2>^{1/2} = 1.143$ km and height $<H> = 1.06$ km for the clouds approximated by paraboloids of rotation. Comparative calculational results are given in Table I.

The differences between the mean fluxes of direct $<S>$ and scattered $<Q_s>$ radiation calculated with the use of the P and PG_1 models reach essential magnitudes (especially in the region of intermediate values of solar zenith angles, where the differences between corresponding values of the probability for the sun to be screened by clouds is maximum). This result seems to be important taking into account the fact that real underlying surfaces are far from being Lambertian and their reflectances depend on the angular distribution of solar radiation at the level of such surfaces. It might be expected that the use of the PG_n models with $n \simeq 5$–6, a random geometric shape of clouds would essentially differ from the paraboloids (see Fig. 5) and the above–indicated differences would be still more pronounced not only for the mean fluxes of transmitted radiation but for the mean albedo as well.

TABLE I.

$\xi^*_\odot$	N	$<S>$		$<G_s>$		$<A>$	
		P	PG_1	P	PG_1	P	PG_1
0	0.130	0.879	0.880	0.069	0.075	0.052	0.046
	0.373	0.633	0.641	0.200	0.204	0.167	0.156
	0.593	0.411	0.403	0.307	0.304	0.281	0.294
30	0.130	0.860	0.854	0.077	0.084	0.063	0.062
	0.373	0.611	0.586	0.208	0.225	0.181	0.189
	0.593	0.388	0.354	0.302	0.330	0.309	0.316
60	0.130	0.769	0.684	0.118	0.181	0.113	0.135
	0.373	0.415	0.281	0.265	0.364	0.320	0.354
	0.593	0.195	0.086	0.336	0.398	0.469	0.516
80	0.130	0.482	0.453	0.225	0.257	0.293	0.290
	0.373	0.085	0.076	0.339	0.335	0.576	0.589

$\xi^*_\odot$ is the zenith solar angle.

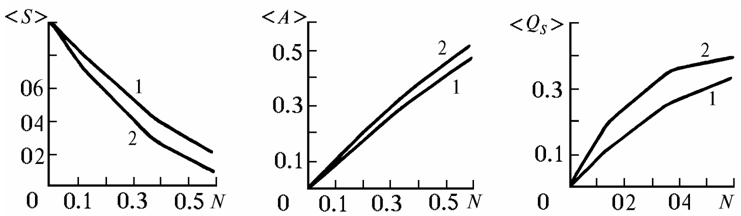

FIG. 5. Dependence of the mean fluxes $<S>$, $<A>$, and $<Q_s>$ on the random geometry of individual clouds at $\xi^*_\odot = 60°$, $\Sigma = 30$ km^{-1}, $<H> = 1.06$ km, and $<D^2>^{1/2} = 1.143$ km (curve 1 refers to the calculations for a Poisson model, curve 2 refers to Poisson–Gaussian PG_1 model).

The results of comparison permit to draw the preliminary conclusion that because of the nonlinear dependence of the radiation field upon the cloud characteristics the random geometric shape of individual clouds can have significant effect on the transfer of solar radiation through cumulus clouds.

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