STATISTICS OF PHOTOELECTRONS OF LASER SIGNALS OBEYING THE LOGNORMAL INTENSITY DISTRIBUTION

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Relations for energy moments of the non–Gaussian radiation field obeying the lognormal intensity distribution which permit one to calculate any moments for a number of photoelectrons for arbitrary ratio between the sampling time and the correlation length of the field have been derived. Analytical approximations for the moments, which refine the well–known asymptotes and depend parametrically not only on the variance of the intensity logarithm, but also on the relative sampling time, have been derived. The range of applicability of these approximations has been determined.

Introduction. In real conditions of propagation of laser radiation through the atmosphere the random field at the receiver may obey the non–Gaussian statistics as a result of violation of the quantum central limit theorem,[1] e.g., due to the following correlations: between the paths of scatterers in the process of single scattering by the turbulence,[2] between the scattering acts in the process of multiple scattering,[3] and between the phases when the beam propagates through the turbulence,[4,5] due to the small number of aerosols in the scattering volume.[2] In these cases the statistics of photoelectrons (PEs) obtained for the Gaussian fields[6] became invalid.

The lognormal approximation of the light intensity distribution has the widest range of applicability for the non–Gaussian field.[4,5,7–9] The statistics of the number n of the PEs appropriate to it has been examined in Refs. 10 and 11 for the sampling period $T \ll \tau_c$ (τ_c is the correlation time of the field). In our paper, this statistics describes the moments for arbitrary ratios between T and τ_c.

"Rigorous" numerical solution. Let $P(t)$ be the received power, $U = \int_t^{t+T} P(t')\mathrm{d}t'$ be the energy received for the period $[t, t+T]$, and m_p and σ_p be the parameters of lognormal distribution P. The "level" $V(t) = \ln P(t)$ is the Gaussian random process with the mean $m_V = \ln m_p - \frac{1}{2}\ln\left(1+\gamma_p^2\right)$, the variance $s_V^2 = \ln\left(1+\gamma_p^2\right)$ and the autocorrelation coefficient $\rho_V(t)$, where $\gamma_p = \sigma_p/m_p$.

In the stationary case, the energy moments are given by the formula

$$\langle U^k\rangle = \exp(km_V) \times$$

$$\times \int_0^T \cdots \int_0^T \exp\left[\frac{\sigma_V^2}{2}\sum_{i=1}^{k}\sum_{j=1}^{k}\rho_V\left(t_i - t_j\right)\right]\mathrm{d}t_1 \cdots \mathrm{d}t_k . \qquad (1)$$

In particular,

$$\langle U^k\rangle \equiv m_U = T\exp\left(m_V + \sigma_V^2/2\right),$$

$$\langle U^2\rangle \equiv 2\exp\left(2m_V\right)\int_0^T (T-\tau)\exp\left\{\sigma_V^2\left[1+\rho_V(\tau)\right]\right\}\mathrm{d}\tau .$$

Using the well–known relation[1] between the factorial moments of the PEs and the initial energy moments

$$\langle n^{[k]}\rangle = \eta^k\langle U^k\rangle$$

(η is the quantum efficiency of the detector), we can write relations for the initial, central, and factorial moments and for the cumulants of the n of arbitrary order, in particular,

$$D(n) = \left(\eta m_p T\right)^2\left[\frac{2}{T}\int_0^T\left(1-\frac{\tau}{T}\right)\times\right.$$

$$\left.\times\exp\left\{\sigma_V^2\left[1+\rho_V(\tau)\right]\right\}\mathrm{d}\tau + \left(\eta m_p T\right)^{-1} - 1\right].$$

In order to calculate the relative energy moments $\langle U^k\rangle/\langle U\rangle^k$, an effective algorithm has been constructed in Ref. 12 which makes use of the symmetries in integrals (1). The numerical calculations of the PEs moments up to 15th order have been performed for two shapes of ρ_V

$$\rho_V^{(a)} = \exp(-t^2/a^2)\ ; \quad \rho_V^{(b)} = \exp(-\tau^2/a^2)\cos(\tau/b)$$

for which

$$\tau_c^{(a)} = \left(\frac{\pi}{2}\right)^{1/2}\cdot\frac{a}{2},\quad \tau_c^{(b)} = \left(\frac{\pi}{2}\right)^{1/2}\cdot\frac{a}{2}\cdot\exp\left(-\xi^2/2\right),$$

where $\xi = a/b$.

Figure 1 shows the relative variance k_v and the assymmetry coefficient k_a of the PEs for both shapes of ρ_V. In the case of recording the laser radiation transmitted through the turbulent atmosphere, the parameter σ_V characterizes the intensity of atmospheric turbulence.[4]

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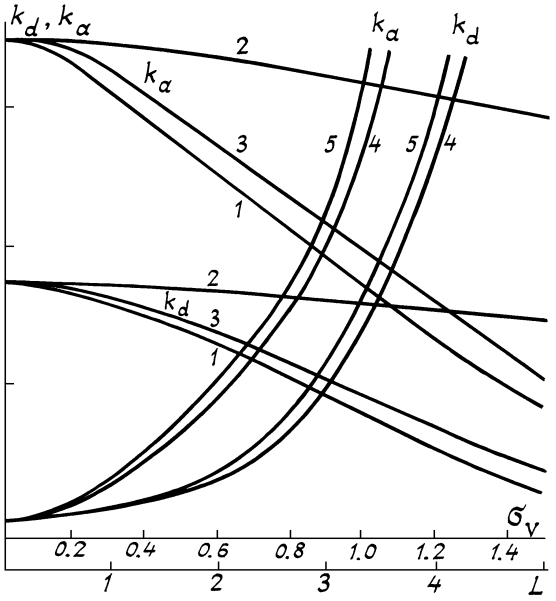

FIG. 1. The dependence of the relation variance k_v and the asymmetry coefficient k_a of the PE distribution on the normalized sampling time L(curves 1, 2, and 3) for σ_V (curves 4 and 5) for $L = 1$. The calculations have been performed for two different models of ρ_V: $\rho_V^{(a)}$ (curve 1 and 4) and $\rho_V^{(b)}$ at $\xi = 2$ *(curves 1 and 5) and* $\xi = 1$ *(curve 3);* $<n> = 1$.

One can see from Fig. 1 that as the number $L = T/\tau_c$ of the time phase elements decreases, k_v and k_a tend to their maxima determined in Refs. 10 and 11 and with increase of L they tend to the values, which correspond to the Poisson distribution.[6] The Poisson approximation of the PE distribution is applicable for $\rho_V^{(a)}$ and $\rho_V^{(b)}$ when L is moderately large and ξ is small and for $\rho_V^{(b)}$ when L is very large and $\xi > 1$. The approximation from Refs. 10 and 11, vice versa, is applicable for $\rho_V^{(b)}$ when L is moderately small and $\xi > 1$ and for $\rho_V^{(a)}$ and $\rho_V^{(b)}$ when L is very small and ξ is small. With increase of σ_V, the values k_v and k_a increase markedly starting from $\sigma_V \sim 0.5$–0.7, which is called the "intermediate" intensity of turbulence in Ref. 5. When $\sigma_V = 0$ the atmosphere does not modulate an amplitude–stabilized laser radiation so that k_v and k_a obey to the Poisson distribution. When $\sigma_V \le 0.3$, the distribution of the PEs can be approximated by the Poisson distribution even for $L \sim 1$. Figure 1 demonstrates that the range of applicability of the well–known analytical approximations for the distribution and the moments of the PEs[10,11] is limited by the values $L \le 0.2$–0.5 and depends on the shape of ρ_V.

Approximate analytic solution. Let us accept the approximation $\rho_V(t) = 1 - t^2/\tau_*^2$, where $\tau_*^2 = -2/\rho_V''(0)$, assuming that $\rho_V(t)$ is double–differentiable function at $t = 0$. In this case we have[13]

$$<U^k> \simeq <U^k>_a = \tau_*^k \exp\left(\sigma_V^2 k^2/2 + k\, m_V\right) I_k ; \qquad (2)$$

where

$$I_k = \int\limits_{-\nu_*/2}^{\nu_*/2} \cdots \int\limits_{-\nu_*/2}^{\nu_*/2} \exp\left(\sigma_V^2\, \mathbf{k}^{\mathrm{T}}\, \mathbf{M}\mathbf{k}\right) \mathrm{d}\mathbf{k} ,$$

$$k \ge 2 ,\ \nu_* = T/\tau_* \le 1/2 ,\ \kappa = \left\{\kappa_i\right\} ,$$

$$\mathbf{M} = \left\{k\delta_{ij} - 1\right\} ,\ i, j = \overline{1, k} .$$

When $k = 2$

$$<U^2>_a = \frac{\tau_*^2}{\sigma_V} \exp\left(2\sigma_V^2 + 2m_V\right) \times$$

$$\times \left[\nu_* \pi^{1/2} \Phi(\nu_*\, \sigma_V) - \frac{1}{\sigma_V}\left(1 - \exp\left(-\nu_*^2\, \sigma_V^2\right)\right)\right] ,$$

where

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \int\limits_0^x \exp\left(-t^2\right) \mathrm{d}t .$$

When $k \ge 3$ the quantity $<U^n>_a$ cannot be expressed in terms of tabulated functions; for this reason, we should approximate I_k. Let us replace the integration variables $\mathbf{y} = \mathbf{u}^{\mathrm{T}}\kappa$, where $\mathbf{u}$ is the orthogonal matrix, diagonalizing $\mathbf{u}^{-1}\mathbf{M}\mathbf{u} = \mathrm{diag}(k, k, \ldots, k, 0)$ the symmetric nonnegatively defined circulant matrix $\mathbf{M}$ with eigennumbers $\lambda_\mu = k$, $\mu = \overline{1, k-1}$, and $\lambda_k = 0$. In so doing, the integration limits remain unchanged. In this case, we have

$$I_k \simeq I_k' = \int\limits_{-\nu_*/2}^{\nu_*/2} \cdots \int\limits_{-\nu_*/2}^{\nu_*/2} \exp\left(-k\, \sigma_V^2 \sum_{\mu=1}^{k-1} y_\mu^2\right) \mathrm{d}\mathbf{y} =$$

$$= \nu_* \left[(\pi/k)^{1/2}\, \sigma_V^{-1}\, \Phi(z)\right]^{k-1} , \qquad (3)$$

where $z = k^{1/2}\sigma_V\, \nu_*/2$.

From the viewpoint of geometry, this approximation means such a rotation of the k–dimensional integration cube that its edges become parallel to the eigenvectors of the matrix $\mathbf{M}$. When $z \ll 1$ this rotation introduces but insignificant change in the value of the integral, i.e., I_k' has a correct asymptote $I_k' \simeq \nu_*^k$. As $z \to \infty$, the equal–value elipsoids of the quadratic form in I_k are subtended to a straight line, on which the eigenvector $\mathbf{b} = \{b_i \equiv \mathrm{const}\}$, $i = \overline{1, k}$, corresponding to $\lambda_k = 0$, lies, so that the contribution to the integral becomes proportional to the length of this straight line segment wich lies inside the integration volume. The rotation decreases its length by a factor of $k^{1/2}$, from the diagonal length $k^{1/2}\nu_*$ to the edge ν_*. For this reason, the asymptote $I_k' = \nu_*[(\pi/k)^{1/2}\sigma_V^{-1}]^{k-1}$ is smaller than the correct value by a factor of $k^{1/2}$ for $z \gg 1$. In order to refine I_k', it can be corrected, e.g., $I_k \simeq I_k' f$, where

$f = 1 + (k^{1/2} - 1)\Phi(a_k z)$, the coefficients a_k, $k = 2, 3, \ldots$, can be determined by numerical methods with the help of Ref. 12.

Applicability of the analytic approximation. Our analysis of the applicability of this approximation of $<U^k>$ based on Eqs. (2) and (3) covers $\rho_V^{(a)}$, $\rho_V^{(b)}$, and the values of τ_* which are determined by the choice of the parabola in the quadratic approximation of ρ_V in terms of the second derivative (the Taylor parabola – T) and based on the equality of the correlation lengths (the Mandel parabola – M). In the case of $\tau_*^{(T)}$ the correlation is deliberately underestimated and in the case of $\tau_*^{(M)}$ only the correlation shape changes for one and the same correlation lengths: $\tau_*^{(T,a)}=2(2/\pi)^{1/2}\tau_c$; $\tau_*^{(T,b)}=2(2/\pi)^{1/2}(1 + \xi^2/2)^{-1/2}\exp(\xi^2/2)\tau_c$; $\tau_*^{(M,a,b)} = (15/8)\tau_c$; where according to Ref. 6

$$\tau_c = \int\limits_0^\infty \rho_V^2(\tau)\, d\tau .$$

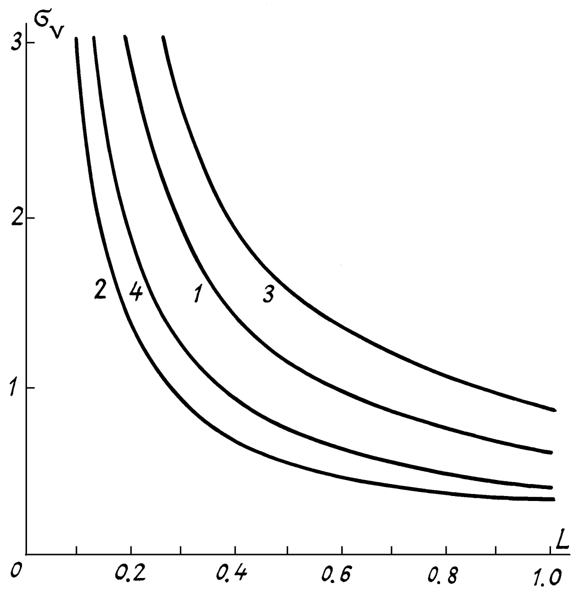

FIG. 2. The countours of the error δ of the analytical approximaton of the energy moments: δ = 5% and k = 5 (1), 10 *(2)*; δ = 10% *and k* = 5 *(3)*, 10 *(4)*.

Figure 2 shows the ranges of applicability of $<U^k>_a$ with I'_k instead of I_k for $\rho_V^{(a)}$ and $\tau_*^{(M)}$. These ranges lie below the given curves and ensure an error $\delta = (<U^k>_a - <U^k>)/<U^k>$ not larger than the prescribed one for all the moments whose orders do not exceed the orders of the moments given above. As expected, the accuracy of the obtained analytical approximation deteriorates with increase of k, L, and σ_V. For $k \le 5$ with $\delta \le 5\%$, $<U^k>_a$ is applicable for weak ($0 < \sigma_V < 0.5$), moderate ($0.5 \le \sigma_V \le 0.9$) and strong ($0.9 < \sigma_V < 1.5$) intensity fluctuations when $L > 1.3$, $0.66 \le L \le 1.3$, and $0.37 < L < 0.666$, respectively. For $k = 2$, d ≤ 5% when $0 < L \le 1$ and $0 < \sigma_V \le 4$.

Conclusion. The obtained statistical characteristics of the PE numbers on the level of the moments are valid for the non–Gaussian fields which obey the lognormal intensity distribution at the point receiving aperture for arbitrary ratio L between the sampling time and the correlation length of the field. These characteristics depend parametrically on L and σ_V, which permits one to analyze the dependence of the PE statistics on the conditions of recording of laser signals and to solve the inverse problems of photoelectron statistics. The analytical approximations proposed for the photoelectron moments have a wider range of applicability compared to the well–known approximations described in Refs. 10 and 11 and are simple in use.

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