Algorithm of reconstructing the phase front based on smoothing two-dimensional normalized cubic B-splines

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Based on mathematical apparatus of smoothing two-dimensional normalized cubic B-splines the algorithm for reconstructing the wave fronts of optical radiation has been synthesized capable of effectively optimizing the nonstationary distortions in the presence of noise in the control channels. The analytical expression has been obtained for selecting the number of elements of the Hartmann sensor optimal in terms of providing a preset accuracy of reconstructing the wave fronts.

The potentialities of optical measuring systems, owing to their high precision, are largely limited by the conditions of propagation of optical waves in actual material media. When propagated through the atmosphere the optical waves are distorted due to fluctuations of the refractive index. One of the effective methods of decreasing the distorting effects of turbulent inhomogeneities is the use of methods of adaptive space-time compensation for the distortions of an optical signal.

For this reason, to compensate for nonstationary distortions of optical radiation, we use the systems where the optical radiation phase is measured at different points of the entrance pupil with the subsequent formation of the wave front distribution over the whole entrance pupil. At present, because of the specific properties of the square-law detection, the Hartmann sensors are used in optics that measure the mean, over a subaperture, tilts of a wave front proportional to the following values:

$$k^{-1}\frac{\partial\varphi(x,\ y)_{i,j}}{\partial x}+n^{x}_{i,j},\ \ k^{-1}\frac{\partial\varphi(x,\ y)_{i,j}}{\partial y}+n^{y}_{i,j}, \qquad (1)$$

where k is the wave number, φ is the function describing the phase disturbance; $n^{x(y)}_{i,j}$ is the output noise of the corresponding channels of a quadrant-type photodetector.

The existing methods of wave front reconstruction[1–3] have some drawbacks, which do not allow achieving high precision of responding the nonstationary distortions.

First, the use of polynomial approximation is less effective than the spline-function approximation since the number of residual loss constants for parabolic polynomials exceed the number of splines of the same power by a factor of 5 and for cubic ones – by a factor of 18.[4] Secondly, these methods were developed without the consideration for the presence of noise in control channels described by the Poisson distribution law.[5] Use of traditional optimal methods of nonlinear filtration based on the Stratonovich multidimensional equation is impossible in this case due to the extremely large bulk of calculations, which gives no way of realization of such algorithms on a real time scale.

It should be stressed that simple algorithms of constructing splines do not always allow one, because of insufficient stability to the rounding-off operation, to obtain high precision of the approximation function. Therefore, to increase the accuracy of wave front reconstruction their local and basis forms should be used, and under conditions of significant measurement errors the use of smoothing splines is most effective allowing to compensate for anomalous outliers of radiation measurements and the effect of noise.

Besides, with the availability of data on the noise intensity or the prediction about the optical radiation propagation along the path, the use of smoothing splines enables one to increase additionally the accuracy by choosing optimal values of the smoothing coefficients. Therefore, to solve the problem on synthesis of the algorithm, it is quite useful to make use of the apparatus of smoothing two-dimensional normalized cubic B-splines of the defect 1.

By these splines one usually[6] means the normalized finite functions defined on a certain finite area-carrier, coinciding on the subintervals formed by a grid with some algebraic polynomials not higher than the third power and doubly continuously differentiable. In this case the normalizing factor equals to the arithmetic mean of the steps on the small area where the B-splines differs from zero.

Choosing the cubic B-splines for solving the above-mentioned problem is based on the following consideration. The structure function of an optical wave phase propagated through the layer of a turbulent medium is proportional to the linear coordinate raised to the 5/3 power and is a continuous function monotonically increasing over the entire domain of definition. Since the exactness of smoothing is mainly determined by the function evenness within the segment between the spline nodes,[6] the choice of the power of the approximating function higher than 3, is inadvisable because it provides only for a slight increase in the

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exactness. This fact is confirmed by the computational experiments made by the authors.

The goal of this work was to synthesize the algorithm of wave front reconstruction based on smoothing two-dimensional normalized cubic B-splines accounting for the presence of noise in the control channels of an adaptive optical system and having high precision characteristics.

Now we set the Hartmann sensor of the size $[ab]\times[cd]$, consisting of $N\times M$ quadrant photodetectors $h_x = R_x/N$, $h_y = R_y/M$ (where R_x, R_y are the sensor size along the corresponding axes of coordinates), each measuring the mean wave front tilts over a subaperture in two perpendicular planes *zox* and *zoy*. The photodetectors are characterized by a pair of indices; i is the number of line and j is the number of column ($i = 1, N$, $j = 1, M$). The measured values of partial derivatives (1) correspond to the middle of the ijth photodetector. Let us introduce, on the plane *xoy*, the set of nodes of splines with uniform steps h_x, h_y along the corresponding coordinate axes

$$\Delta_x : x_0 < x_1 = a < x_2 < \ldots < x_N = b < x_{N+1},$$

$$\Delta_y : y_0 < y_1 = c < y_2 < \ldots < y_M = d < y_{M+1}. \quad (2)$$

Considering the characteristics of measurements, carried out by quadrant photodetectors, the phase value at the splines nodes can be expressed through its partial derivatives

$$\varphi(x,y)_{i,j} = \frac{k^{-1}}{2}\frac{\partial\varphi(x,\,y)_{i,j}}{\partial x}\,h_x + \frac{k^{-1}}{2}\frac{\partial\varphi(x,\,y)_{i,j}}{\partial y}\,h_y. \quad (3)$$

In this case the wave front sought can be synthesized as a system of smoothing two-dimensional normalized cubic B-splines of the defect 1 (Ref. 6)

$$S(x,\,y) = \sum_{j=0}^{M}\sum_{i=0}^{N} b_{i,j}\,B_3^{i,j}(x,\,y), \quad (4)$$

where $b_{i,j}$ are the coefficients of splines, $B_3^{i,j}(x,\,y)$ is the third-power B-splines, corresponding to the ijth area.

To reduce the formula, we denote $B_3^{i,j}(x,\,y)$ as $B_3^{i,j}$, then in the general form the expression for B-splines can be written as

$$B_n^{i,j} = \frac{x - x_i}{x_{i+n} - x_i}\,\frac{y - y_j}{y_{j+n} - y_j}\,B_{n-1}^{i,j} + \frac{x_{i+n+1} - x}{x_{i+n+1} - x_{i+1}}\,\frac{y - y_j}{y_{j+n} - y_j}\,B_{n-1}^{i+1,j} +$$
$$+ \frac{x - x_i}{x_{i+n} - x_i}\,\frac{y_{j+n+1} - y}{y_{j+n+1} - y_{j+1}}\,B_{n-1}^{i,j+1} +$$
$$+ \frac{x_{i+n+1} - x}{x_{i+n+1} - x_{i+1}}\,\frac{y_{j+n+1} - y}{y_{j+n+1} - y_{j+1}}\,B_{n-1}^{i+1,j+1}, \quad (5)$$

where n is the power of the splines.

After simple, but cumbersome calculations the expression for the cubic B-splines takes the form

$$B_3^{i,j} = f_x\,f_y\,B_0^{i,j} + g_x\,f_y\,B_0^{i+1,j} + f_x\,g_y\,B_0^{i,j+1} +$$
$$+ g_x\,g_y\,B_0^{i+1,j+1} + p_x\,f_y\,B_0^{i+2,j} + f_x\,p_y\,B_0^{i,j+2} +$$
$$+ p_x\,g_y\,B_0^{i+2,j+1} + g_x\,p_y\,B_0^{i+1,j+2} + p_x\,p_y\,B_0^{i+2,j+2} +$$
$$+ w_x\,f_y\,B_0^{i+3,j} + f_x\,w_y\,B_0^{i,j+3} + w_x\,g_y\,B_0^{i+3,j+1} +$$
$$+ g_x\,w_y\,B_0^{i+1,j+3} + w_x\,p_y\,B_0^{i+3,j+2} +$$
$$+ p_x\,w_y\,B_0^{i+2,j+3} + w_x\,w_y\,B_0^{i+3,j+3}; \quad (6)$$

$$f_{x(y)} = \frac{1}{6}\,(2 + \chi_{x(y)})^3;\; g_{x(y)} = \left(\frac{2}{3} - \chi^2_{x(y)} - \frac{\chi^3_{x(y)}}{2}\right);$$

$$p_{x(y)} = \left(\frac{2}{3} - \chi^2_{x(y)} + \frac{\chi^3_{x(y)}}{2}\right);\; w_{x(y)} = \frac{1}{6}\,(2 - \chi_{x(y)})^3;$$

$$\chi_x = \frac{(x - x_{i+2})}{h_x},\; \chi_y = \frac{(y - y_{j+2})}{h_y};$$

$$B_0^{i,j} = \begin{cases} 1 \text{ at } x \in [x_i;\, x_{i+1}] \times [y_j;\, y_{j+1}], \\ 0 \text{ at } x \notin [x_i;\, x_{i+1}] \times [y_j;\, y_{j+1}]. \end{cases}$$

The problem of smoothing will be solved by minimizing the functional[6]

$$J = \int_a^b \int_c^d [D^{2,2}\,S(x,\,y)]^2\,\mathrm{d}x\,\mathrm{d}y +$$
$$+ \rho^{-1} \sum_{i=0}^{N} \int_c^d [D^{0,2}\,S(x_i,\,y)]^2\,\mathrm{d}y +$$
$$+ \omega^{-1} \sum_{j=0}^{M} \int_a^b [D^{2,0}\,S(x,\,y_j)]^2\,\mathrm{d}x +$$
$$+ (\rho\omega)^{-1} \sum_{i=0}^{N}\sum_{j=0}^{M} [S_{i,j} - \varphi_{i,j}]^2, \quad (7)$$

where ρ and ω are the smoothing coefficients; $S_{i,j}$, $\varphi_{i,j}$ are the values of splines and the wave front phase at the nodes of the coordinate system, $D^{\alpha,\beta} = \dfrac{\partial^{\alpha+\beta}\,S(x,y)}{\partial x^\alpha\,\partial y^\beta}$.

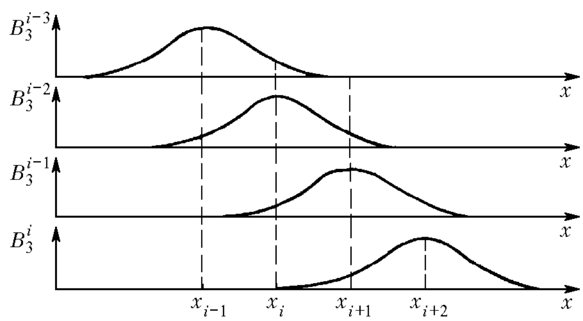

Fig. 1. The jth cross section of the B-splines carriers.

Now we consider an area $[x_i;\, x_{i+1}]\times[y_j;\, y_{j+1}]$ and define the coefficients of splines at the center of the corresponding carrier function (Fig. 1), then for this area we can write

$$S_{i,j} = f_x f_y b_{i+2,j+2} + g_x f_y b_{i+1,j+2} + f_x g_y b_{i+2,j+1} + g_x g_y b_{i+1,j+1} + p_x f_y b_{i,j+2} + f_x p_y b_{i+2,j} + p_x g_y b_{i,j+1} + g_x p_y b_{i+1,j} + p_x p_y b_{i,j} + w_x f_y b_{i-1,j+2} + f_x w_y b_{i+2,j-1} + w_x g_y b_{i-1,j+1} + g_x w_y b_{i+1,j-1} + w_x p_y b_{i-1,j} + p_x w_y b_{i,j-1} + w_x w_y b_{i-1,j-1}. \quad (8)$$

Now we introduce normalized coordinates for splines equal to $\chi_x = (x - x_i)/h_x$, $\chi_y = (y - y_j)/h_y$ on the axes ox and oy, respectively. After simple arithmetical transformations, and grouping relative to the coefficients of splines, and substituting in Eq. (4) we obtain an analytical expression for the splines

$$S(x,y) = \sum_{i=1}^{N-1}\sum_{j=1}^{M-1} \frac{\chi_y^3\chi_x^3}{36}(F_{j+2} - 3F_{j+1} + 3F_j - F_{j-1}) + \frac{\chi_y^3\chi_x^2}{12}(G_{j+2} - 3G_{j+1} + 3G_j - G_{j-1}) + \frac{\chi_y^3\chi_x^1}{12}(H_{j+2} - 3H_{j+1} + 3H_j - H_{j-1}) + \frac{\chi_y^3}{36}(L_{j+2} - 3L_{j+1} + 3L_j - L_{j-1}) + \frac{\chi_y^2\chi_x^3}{12}(F_{j+1} - 2F_j + F_{j-1}) + \frac{\chi_y^2\chi_x^2}{4}(G_{j+1} - 2G_j + G_{j-1}) + \frac{\chi_y^2\chi_x}{4}(H_{j+1} - 2H_j + H_{j-1}) + \frac{\chi_y^2}{12}(L_{j+1} - 2L_j + L_{j-1}) + \frac{\chi_y\chi_x^3}{12}(F_j - F_{j-1}) + \frac{\chi_y\chi_x^2}{4}(G_j - G_{j-1}) + \frac{\chi_y\chi_x}{4}(H_j - H_{j-1}) + \frac{\chi_y}{12}(L_j - L_{j-1}) + \frac{\chi_x^3}{36}(F_{j+1} + 4F_j + F_{j-1}) + \frac{\chi_x^2}{12}(G_{j+1} + 4G_j + G_{j-1}) + \frac{\chi_x}{12}(H_{j+1} + 4H_j + H_{j-1}) + \frac{1}{36}(L_{j+1} + 4L_j + L_{j-1}), \quad (9)$$

where

$$F_* = (b_{i+2,*} - 3b_{i+1,*} + 3b_{i,*} - b_{i-1,*});$$

$$G_* = (b_{i+1,*} - 2b_{i,*} + b_{i-1,*});$$

$$H_* = (b_{i+1,*} - b_{i-1,*});\ L_* = (b_{i+1,*} + 4b_{i,*} + b_{i-1,*}).$$

In this case the minimized functional is written as

$$J = \sum_{i=1}^{N-1}\sum_{j=1}^{M-1}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}} [D^{2,2} S(x,y)]^2 \,dx\,dy + \rho^{-1}\sum_{i=0}^{N}\sum_{j=1}^{M-1}\int_{y_j}^{y_{j+1}} [D^{0,2} S(x_i,y)]^2 \,dy + \omega^{-1}\sum_{j=0}^{M}\sum_{i=1}^{N-1}\int_{x_i}^{x_{i+1}} [D^{2,0} S(x,y_j)]^2 \,dx + \frac{(\rho\omega)^{-1}}{36}\sum_{i=0}^{N}\sum_{j=0}^{M} [(L_{j+1} + 4L_j + L_{j-1}) - \varphi_{i,j}]^2. \quad (10)$$

Calculating the integrals and conducting the transformations, complying with the dimensionality of the functional components, we obtain

$$J = \frac{1}{18}\sum_{i=1}^{N-1}\sum_{j=1}^{M-1}(2V_1^2 + 6W_1^2 + 6V_2^2 + 18W_2^2 + 6V_1W_1 + 9V_1W_2 + 6V_1V_2 + 9V_2W_1 + 19W_2W_1 + 18V_2W_2) + \frac{\rho^{-1}}{108}\sum_{i=0}^{N}\sum_{j=1}^{M-1}(3U_1U_2 + 3U_2^2 + U_1^2) + \frac{\omega^{-1}}{108}\sum_{j=0}^{M}\sum_{i=1}^{N-1}(3V_3W_3 + 3W_3^2 + V_3^2) + \frac{(\rho\omega)^{-1}}{36}\sum_{i=0}^{N}\sum_{j=0}^{M}[(L_{j+1} + 4L_j + L_{j-1}) - \varphi_{i,j}]^2; \quad (11)$$

$$V_1 = (F_{j+2} - 3F_{j+1} + 3F_j - F_{j-1}),$$

$$V_2 = (F_{j+1} - 2F_j + F_{j-1}),\ V_3 = (F_{j+1} + 4F_j + F_{j-1});$$

$$W_1 = (G_{j+2} - 3G_{j+1} + 3G_j - G_{j-1}),$$

$$W_2 = (G_{j+1} - 2G_j + G_{j-1}),\ W_3 = (G_{j+1} + 4G_j + G_{j-1});$$

$$U_1 = (L_{j+2} - 3L_{j+1} + 3L_j - L_{j-1}),$$

$$U_2 = (L_{j+1} - 2L_j + L_{j-1}).$$

To find the coefficients of splines (9), that minimize the functional (11), it is necessary to calculate its partial derivatives for every coefficient and to equate them with zero. As a result we obtain the system of $(N + 2)$ by $(M + 2)$ linear equations of the form $QA = Z$. The matrix of coefficients Q has a clearly defined diagonal form and well conditioned. When solving this system of equations using one of the known methods,[6] we calculate the values of unknown coefficients.

In solving practical problems, we need, as a rule, to realize the wave front reconstruction with a preset accuracy. This is achieved by selecting a step of a splines grid taking into account the characteristics of the smoothing function and the corresponding estimates of the smoothing error. Since the step size is strictly bound with the sizes of a quadrant-type photodetector, this enables one to select the number of sensor elements optimal from the viewpoint of providing the preset accuracy of the wave front reconstruction. Considering the local characteristics of phase perturbations due to the dimensions of inhomogeneous medium of propagation, the size of the grid step, providing the minimum error of reconstruction over the corresponding coordinates, can be determined from the following expressions[6]:

$$h(x) = \sqrt{\frac{8\varepsilon}{\| D^{2,0}\, S(x,y) \|_{[x_i,x_{i+1}]}}}\,,$$

$$h(y) = \sqrt{\frac{8\varepsilon}{\| D^{0,2}\, S(x,y) \|_{[y_j,y_{j+1}]}}}\,, \qquad (12)$$

where ε is the error of determination of the splines; $\|D^{2.0}S(x, y)\|_{[x_i,x_{i+1}]}$, $\|D^{0.2}S(x, y)\|_{[y_i,y_{i+1}]}$ are the norms of the second partial derivatives of the splines in the corresponding intervals.

The peculiarities of the Hartmann sensor design make the application of irregular grid too difficult, therefore an expression for optimal number of sensor elements can be written as

$$N\,M = R_x \left(\frac{8\varepsilon}{\| D^{2,0}\, S(x,y) \|_{[a,b]}} \right)^{-1/2} \times$$
$$\times R_y \left(\frac{8\varepsilon}{\| D^{0,2}\, S(x,y) \|_{[c,d]}} \right)^{-1/2}. \qquad (13)$$

Thus, Eq. (13) defines the number of elements of the Hartmann sensor, necessary for providing the preset accuracy of reconstructing of the wave front.

Thus, based on the Eq. (9) obtained for the smoothing two-dimensional normalized cubic B-splines, we have developed an algorithm of reconstruction of the optical radiation wave front in the presence of noise in the control channels.

It has been found that Eq. (13) for selecting the number of elements of the Hartmann sensor is optimal for achieving the preset accuracy of compensation for the nonstationary distortions.

The proposed algorithm of wave front reconstruction was realized in the MATHCAD-7 PRO for the case of $N = M = 5$.

The numerical experiment has shown that the application of mathematical apparatus of smoothing two-dimensional normalized cubic B-splines enables one:

– to efficiently consider for a stochastic signal nature at the output of a Hartmann sensor;

– if *a priori* information about the character and intensity of noise is available, it is necessary to operatively introduce a correction in the reconstruction algorithm providing for an additional improvement of the quality of responding the nonstationary distortions. In this case the accuracy of reconstruction is determined by the error of approximating the function $O(h^{n+1})$ and slightly decreases under conditions of an intense noise.

In the computer experiment we tested the effect of selecting the coefficients of smoothing and breakdown of separate quadrant photodetectors on the quality of functioning of the proposed algorithm. Figure 2 shows the calculated results on the error of approximation of the wave front at high and low signal-to-noise ratios at the output of the Hartmann sensor. Analysis of variation of smoothing coefficients indicates that the use of ρ and ω ≤ 1 results in degeneration of the smoothing splines to the interpolation one. In this case the smoothing splines, even in cases of low signal-to-noise ratios, provides for a 1.7 increase of the accuracy as compared with the interpolation splines. In case of a high SNR the selection of optimal values of smoothing coefficients within the limits from 3 to 10 provides for achieving the best accuracy of the approximation.

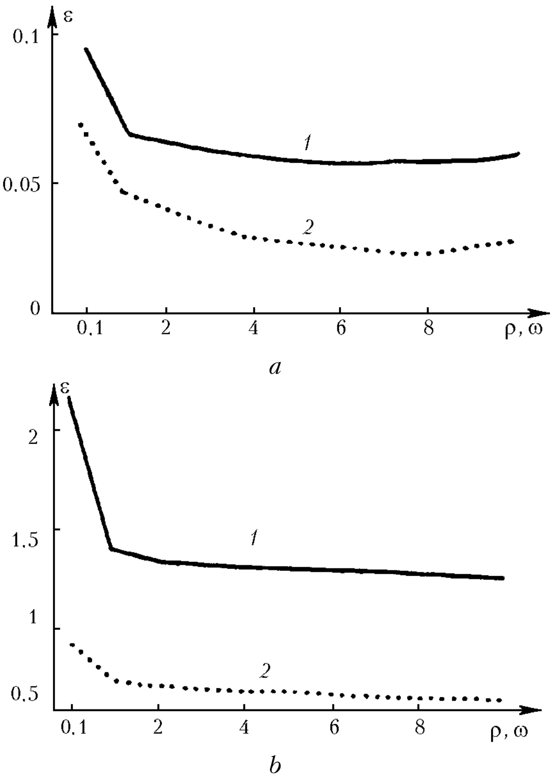

Fig. 2. The dependence of the error of wave front approximation on the smoothing coefficient at low (*a*) and high (*b*) SNR: (*a*) SNR = 0 (curve *1*) and 10 dB (curve *2*); (*b*) SNR = 25 (curve *1*) and 30 dB (curve *2*).

In this case the absence of measurements from several quadrant photodetectors practically does not affect the quality of reconstruction.

In conclusion, it should be noted that the proposed algorithm can be used in calculations and processing of other spatially-distributed parameters, and the method of calculation of smoothing two-dimensional normalized cubic B-splines can be extended to calculation the splines of higher orders.

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