ON THE CONDITIONS OF THE PARABOLIC APPROACH APPLICABILITY

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Applicability of the parabolic approach (PA) to solution of field problems near the geometric focus is considered. A version of quantitative criterion is given. It was shown that the discrepancy between the exact solution and the PA: a) generally speaking is always present that is connected with the pemature truncation of series for the phase of the Green's function and boundary field; b) is unessential if the boundary field and the Green's function has a spherical wavefront; c) can reach considerable value if that and/or other wavefront becomes nonspherical; and, d) is the larger the more characteristic boundary and focal size differ due to beam focusing.

1. INRODUCTION

The parabolic approach (PA), which uses parabolic equations instead of the hyperbolic ones that are typically used in the problems of wave propagation has become very popular in atmospheric optics.[1,2] The same approach is well known in the diffraction theory,[3] nonlinear optics,[4,5] electrodynamics of turbulent media,[6,7] in analysis of Gaussian beams, and in quasioptics.[8] The main idea of the approach comes from an earlier work,[9] and its mathematical aspects are excellently discussed in Ref. 3. A very close subject, i.e., a detailed analysis[10] and some results of it are demonstrated in Section 2. Introduction of the PA itself is possible by different methods: such as comparison with test solutions,[3] transformation of the Green's function,[6] and separation out the oscillating factor,[4] and the like.

Having such a vast literature on the problem it could seem that no problems can arise on the conditions of the PA applicability. However, as it becomes clear now, the situation occurring in the vicinity of convergence of geometrical optics rays (caustic, focus) has to be drastically revised.

2. SHORT REVIEW OF THE PROBLEM

Let the wave equation (for monochromatic case) of hyperbolic type be the initial equation

$$\Delta\Psi(\mathbf{r}) + k^2m^2\Psi(\mathbf{r}) = 0, \tag{1}$$

where $k = 2\pi/\lambda$, $m(\mathbf{r})$ is the refractive index at the point $\mathbf{r}$, and the scalar version is accepted only for simplicity. Transition from Eq. (1) to an approximate equation is performed according to almost standard scenario. Assume that

$$\Psi(\mathbf{r}) = U(\mathbf{r})\exp[ikS(\mathbf{r})] \tag{2}$$

and after substitution of Eq. (2) into Eq. (1) it becomes

$$\Delta U + 2ik\mathrm{grad}U\mathrm{grad}S + ikU\Delta S + k^2m^2U - k^2U(\mathrm{grad}S)^2 = 0. \tag{3}$$

Of course, there are no formal prohibition to write Eq. (2), but it is necessary "to say something" about, for example, S, otherwise Eq. (3) will have two unknown functions. Born and Wolf[11] consider that $\exp(ikS)$ "involves most oscillating part of Ψ", i.e., U is more slow (compared to exp(ikS)) function. Therefore, the relations

$$(\mathrm{grad}S)^2 = m^2,\ \mathbf{n} = \frac{1}{m}\mathrm{grad}S,\ \mathbf{n}\text{ is ort} \tag{4}$$

defining the S value and assumptions on the validity of conditions

$$|\Delta S| \ll \frac{1}{L}\,|\mathrm{grad}S|,\ \left|\frac{\partial^2 U}{\partial l^2}\right| \ll k\left|\frac{\partial U}{\partial l}\right| \tag{5}$$

seem to be reasonable. In Eq. (5) $\partial/\partial l$ is the derivative along the direction $\mathbf{n}$, and L is the distance at which U noticeably changes (i.e. $|\mathrm{grad}U| = 0(U/L)$).

Really relations (4) eliminates the summand $\sim k^2$ (the most explicit source of oscillations) from Eq. (3). The physical meaning of Eq. (4) is well known, i.e., there appears eikonal S and a wave propagates along $\mathbf{n}$ perpendicularly to the surface $S = \mathrm{const}$. It is assumed, of course, that Eq. (4) has the unique solution.

Now, Eqs. (3), (4), and (5) yield

$$\Delta_\perp U + 2ikm\frac{\partial U}{\partial l} = 0 \tag{6}$$

of the parabolic type and $\Delta_\perp$ is the Laplacian of "transverse" relative to the $\mathbf{n}$ coordinates.

Mathematical arranging of similar considerations can be associated with the integral equation equivalent to Eq. (1) (see, for example, Refs. 11 and 12). After its substitution Eq. (2) reduces the investigation of Ψ to integral

$$\int d\rho d\mathbf{q}\Gamma(k, \mathbf{q})\left[m^2(\mathbf{r}-\rho) - 1\right]U(\mathbf{r}-\rho)\times$$

$$\times\exp[ikS((\mathbf{r}-\rho)]\exp(i\mathbf{q}\rho)\ . \tag{7}$$

Here, Γ is the Fourier transform of the Green's function of the Helmholtz operator (Eq. (1) for $m = 1$ and $q = 0(\mathrm{k})$), V is the region where $m \neq 1$, and, of course, $k\sqrt[3]{V} \gg 1$ is the necessary condition of the approach. The latter provides the effectiveness of the asymptotic estimation (7).

Let us first assume, that Eq. (4) has one solution (this justifies Eqs. (2) and (7) and fulfilment of conditions (5)). The first of conditions (5) allows one to reduce $S(\mathbf{r}-\rho)$ to a linear function of ρ, when estimating by Eq. (7), and

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relations (4) enables one to prove the fact that the first derivative of the function in the exponent has no zeros inside the V region. The second of conditions (5) confirms that U has no points of branching (of "pole" or "zero" types). Therefore, Eq. (7) is estimated as a Fourier integral, and

$$U = \sum_{n=0} \xi^n U_n(\mathbf{r}),\ \xi = 1/ik \tag{8}$$

becomes a regular function along ξ.

However, if one assumes Eq. (8) or, more generally, the regularity of Ψ along ξ then the standard theorems from the theory of analitical functions[13] will confirm the uniqueness and finity of S and U. The regularity of Ψ gives a possibility of equalizing coefficients at equal degrees of ξ. This, in turn, yields both Eq. (4) and the recursion chain

$$2\mathrm{grad}\, U_n\, \mathrm{grad} S + U_n \Delta S + \Delta U_{n-1} = 0, \qquad U_{-1} = 0, \tag{9}$$

whose initial step ($n = 0$) corresponds to geometrical optics.

Therefore, the necessary and sufficient condition of existence of Eqs. (2) and (3) is the regularity of Ψ. This by no means guarantees the validity of Eq. (6). but Eqs. (2) and (3) are make the basis for an asymptotic analysis, formally as $k \to \infty$, leading to Eq. (6). Pragmatic significance of such an action is quite clear. Actually, one could hardly hope to proceed far from the initial $n = 0$ in Eq. (9) while it is well known that Eq. (6) is more accurate than the geometric optics approach.

The other side of the discussed subject (according to the meaning of terms necessity and sufficiency) is underlined by understanding that regularity of Ψ disappears when conditions (5) are violated. Really, now one must write, when estimating by Eq. (7), in the argument of the (exp) the function of squared (at least) ρ and make choice of the method of steepest descent that will drastically change Eq. (8). First, this leads to appearance of fractional degrees of ξ and the same will happen when the second of conditions (5) is violated. Second, the presence of two equivalent points on the saddle make the eikonal ambiguous and, therefore, one must write

$$\Psi = U_1 \exp(ikS_1) + U_2 \exp(ikS_2) \tag{10}$$

instead of Eq. (2). It is clear now that substitution of Eq. (10) into Eq. (1) by no means can lead to Eq. (6).

This statement may be interpreted in terms of differential geometry.[10] As it can be shown there, curvilinear coordinates exist, i.e., the eikonal S itself and the polar components (β and γ) of the ort $\mathbf{n}$. The corresponding axes are in the plane tangent to the surface $S = \mathrm{const}$. It is absolutely evident from Eq. (4) that the condition of S unambiguity is $J \equiv D(\mathbf{r})/D(S, \beta, \gamma) \neq 0$ for the Jacobian of transformation of the Cartesian coordinates to the curvilinear ones. The same condition provides unambiguity and finiteness of U_n. As it follows from Eq. (9), $U_n \sim J^{-1/2-\sigma_n}$ (when $\sigma_n > 0$). In the vicinity of $J = 0$ naturally S losses its unambiguity, and therefore one is forced to use the version (10).

Thus, the mathematical equivalence of the conditions $J \neq 0$ and "regulariry of Ψ on ξ" becomes clear. The constructive aspect of this situation is that $J \neq 0$ where $\mathrm{div}(m\mathbf{n})$ cannot be large, i.e., at points far from caustics and focuses. Relationship of the latter regions, physical and mathematical, to the value of $\mathrm{div}(m\mathbf{n})$ is quite evident.[10,11,14]

Thus, the previous analysis has evidenced that Eq. (6) cannot be treated as an approximation of Eq. (1) at points of convergence of geometric optics rays. In fact, this circumstance is repeatedly and persistently underlined in Ref. 3. But, certainly, it should be noted that Eq. (6), written "by itself" has some solutions at points of the rays convergence (they are not peculiar for Eq. (6); the same follows from the differential geometry analysis), and, moreover, the solution structure underlines its explicitly "diffractional" origin ("diffusion" into the region of geometrical shadow, "neck" near the focus and the like). Probably, just this creates an illusion of versatality of the parabolic approach.

However, a possibility of applying the boundary conditions in such a specific way (expansion of the Green's function and so on) that could result in quite correct numerical results. This approach is excellently demonstrated in Ref. 3, where equations of the parabolic type are used for investigation of diffraction on the absolutely reflecting objects.

This quite general (and, in a way, preliminary) considerations define more precisely and illustrate the problems from Sections 3 and 4, and summary of the whole discussion is given in Section 5.

3. SPHERICAL FOCUSING INTO HOMOGENIOUS MEDIUM

Let $z = 0$ be the interface between two media. At $z < 0$ $m_1 = 1$ and for $z > 0$ $m_2 \equiv M > 1$. According to Kirchhoff–Helmholtz theorem for all $z_0 \gg 1/k$ [Ref. 4]

$$\Psi(\mathbf{r}_0) = -\frac{ikz_0 m}{2\pi} \int_{-\infty}^{+\infty}\int A(x, y) \frac{\exp(ikS)}{R^2}\, \mathrm{d}x\mathrm{d}y\ . \tag{11}$$

Change of amplitude A at the interface is neglected and the field at $z = 0$ (in the first medium) is assumed to be a converging spherical wave

$$\Psi \Big|_{z=0} = A(x, y)\exp(-ikR_f)\ . \tag{12}$$

In Eqs. (11) and (12) $\mathbf{r}_0(x_0, y_0, z_0)$ is the observation point; $S = mR - R_f$, $R = \sqrt{z_0^2 + r^2}$, $R_f = \sqrt{f^2 + \rho^2}$, $r^2 = (x - x_0)^2 + (y - y_0)^2$, $\rho^2 = x^2 + y^2$, and f is physically analogous to focal length.

A standard condition

$$r/z_0 \sim \rho/f \ll 1 \tag{13}$$

allows one to write

$$R \simeq z_0 + \frac{r^2}{2z_0} - \frac{r^4}{8z_0^3} + \dots\ , \qquad R_f \simeq f^2 + \frac{\rho^2}{2f} - \frac{\rho^4}{8f^3} + \dots \tag{14}$$

where the first two terms correspond to the PA.

Let us consider the field only on the optical axis ($x_0 = y_0 = 0$ and $\rho = r$). Then passing to polar coordinates ρ and θ, as well as separating out the large value $(mz_0 - f)$ from S and assuming the condition $(1/R^2 \simeq 1/z_0^2)$ holds quite accurately one obtains from Eq. (11) the variant which we assume the exact solution

$$\Psi(z_0) = -\frac{ikm}{2\pi z_0}\exp[ik(mz_0 - f)] \times$$

$$\times \int_0^{2\pi}\int_0^{\infty} A(\theta,\ \rho)\exp(ikS)\rho d\rho d\theta\ , \qquad (15)$$

where $S \equiv S_t = m\sqrt{z_0^2 + \rho^2} - \sqrt{f^2 + \rho^2} - (mz_0 - f)$. The approximate solutions differ from that given by formula (15) only by the form of S. Thus, the parabolic approach is represented by

$$S \equiv S_p = S_1\rho^2,\ S_1 = \frac{1}{2}\left(\frac{1}{f} - \frac{m}{z_0}\right)\ . \qquad (16)$$

For the subsequent approximation from series (14)

$$S \equiv S_m = S_1\rho^2 + S_2\rho^4,\ S_2 = \frac{1}{8}\left(\frac{1}{f^3} - \frac{m}{z_0^3}\right)\ . \qquad (17)$$

In general, solutions (15)–(17) can be investigated only numerically and since $\exp(ikS)$ oscillates quite rapidly, the methods themselves and programs for their execution needs for verification.

In a sufficiently general case of $A(\theta,\ \rho) \sim \exp(-\rho^4/a^4)$ the amplitude of the field (15) and S defined by expression (17) can be analitically represented by

$$|\Psi(z_0)| \equiv A_0 = \frac{km}{4z_0}\sqrt{\frac{\pi a^4}{\sqrt{1 + (kS_2a^4)^2}}}\ |\omega(z)|\ , \qquad (18)$$

where $\omega(z) = \exp(-z^2)\left[1 + \frac{2i}{\sqrt{\pi}}\int_0^z \exp(t^2)dt\right] = \exp(-z^2)\mathrm{erfc}(-iz)$ is well–known function [15] of complex argument

$$z = -kS_1a^2/2\sqrt{1 - ikS_2a^4}\ .$$

Note that the phases of solutions are not considered here just for reasons of saving room.

In the PA (i.e., for S from formula (16)) the solution is obtained from relation (18) at $S_2 = 0$.

Dependences of (z_0) on z_0 values in the vicinity of $z_0 = mf$ computed by direct integration of Eq. (15) for three types of S (formulas (15)–(17)), are shown in Fig. 1. Dots in this figure show the amplitude values obtained from relation (18) using tables and the values S given by expressions (16) and (17). Errors of calculations by both methods do not exceed 1 per cent. In fact there no any noticeable difference between the curves calculated using $S = S_t$ and $S = S_m$.

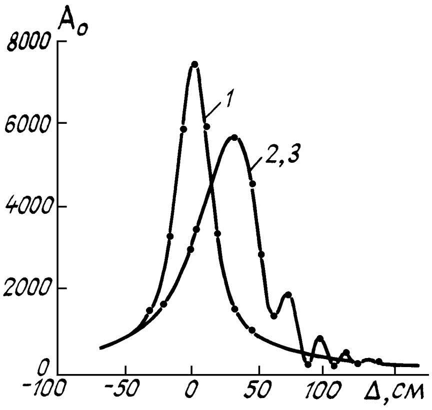

FIG. 1 Dependence of the axial amplitude A_0 of a beam on the value $\Delta = (z_0 - mf)\cdot 1000$. *1) $S = S_p$ (formula (16)), 2) $S = S_m$ (formula (17)), and 3) $S = S_t$ (formula (15)). Amplitude of the boundary field* $\sim\exp(-\rho^4/a^4)$, $a = 2$ cm, $f = 30$ cm, and $m = 1.5$. *The dots show the values obtained from Eq. (18).*

Some results of numerical computer simulations are presented in two next figures. A comparison of the exact solution and the PA obtained for varying geometrical divergence $\alpha = a/f$ of the boundary field is given in Fig. 2. An example of calculations made for a = const is shown in Fig. 3. The computations have been done for a Gaussian amplitude $A \sim \exp(-\rho^2/a^2)$ relevantly corrected for the change of a beam size at focus.

Figures 1 and 3 quite clearly show the situations in which the parabolic approximation is inapplicable, of course, we mean a description of the field at focus. It is evident that the criterion of the PA quality is

$$\kappa = k\rho_{\max}^4|S_2| \ll k\rho_{\max}^2\ |S_1|\ .$$

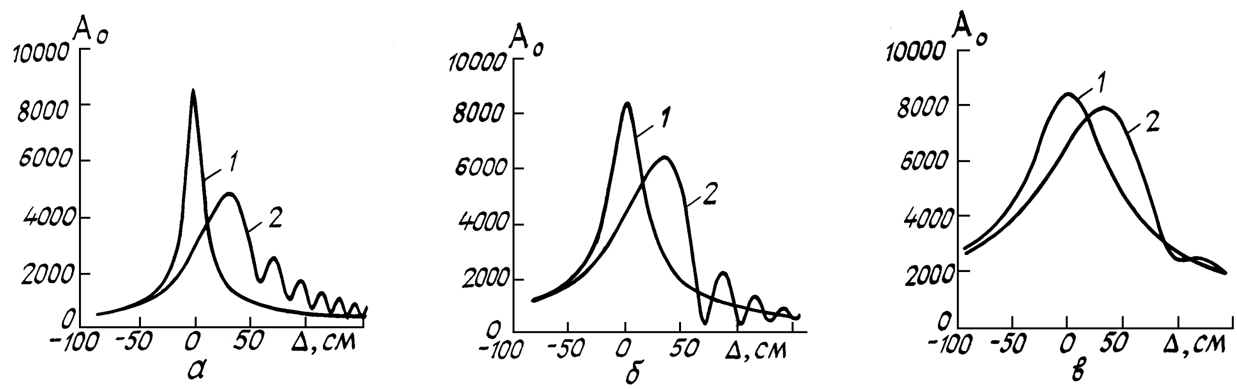

FIG. 2. Dependence of amplitude A_0 on the value Δ (1) PA and 2) exact solution) at different values of the geometrical divergence $\alpha = a/f$: $\alpha = 2/30$ *(a)*, $2/45$ *(b), and* 2 cm/70 cm *(c). The boundary beam is of Gaussian type and* $m = 1.5$.

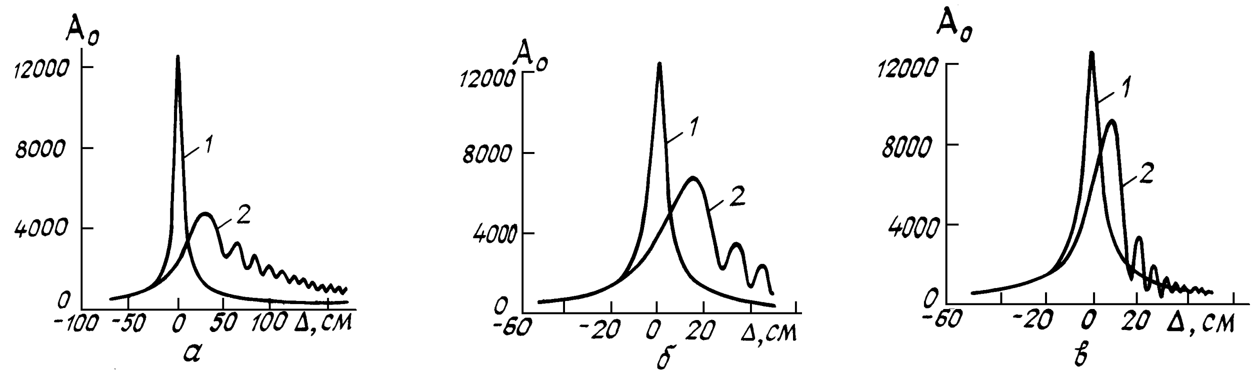

FIG. 3 Dependence of amplitude A_0 on the value Δ (1) PA and 2) exact solution) at a fixed geometrical divergence $\alpha = (a/n)/(f/n) = 2/20$: n = 1 (a), 3 (b), and 9 (c). The boundary beam is of Gaussian type and m = 1.5.

Within a small region around the geometrical focus

$$\kappa \to \kappa_f \sim k\alpha^3 a(1 - 1/m^2) \ll 1 \ . \tag{19}$$

The condition $m \neq 1$ becomes principle what excellently agrees with the discussion in Section 2.

4. CONVERGENT ELLIPTICAL BEAMS IN AN ANISOTROPIC MEDIUM

Solution of this problem is reduced to taking the integrals of the type (15) with the function

$$S = \sqrt{z_0^2 + bx^2 + dy^2} - \sqrt{f^2 + lx^2 + ty^2} \ .$$

Again, $x_0 = y_0 = 0$, constants b and d determine the medium anysothropy, and l and t describe the ellipticity of the boundary field. If a spherical wave is focused into a uniaxial medium with the main optical axis along the x axis, then $d = l = t = 0$ and in an approximation, similar to (17), we have

$$S = -S_1x^2 + S_2x^4 + S_3x^2y^2 + \tilde{S}(y),$$

where $S_1 = \frac{1}{2}\left(\frac{1}{f} - \frac{b}{z_0}\right)$, $S_2 = \frac{1}{8}\left(\frac{1}{f^3} - \frac{b^2}{z_0^3}\right)$, $S_3 = \frac{1}{4}\left(\frac{1}{f^3} - \frac{b}{z_0^3}\right)$,

$$\tilde{S}(y) = \frac{y^2}{2}\left(1/f - 1/z_0\right) + \frac{y^4}{8}\left(1/f^3 - 1/z_0^3\right) \ .$$

Such a structure of S gives rise to two focuses: at $z_0 \sim bf$ there is minimum beam size along the x_0 axis and at $z_0 \sim f$ along the y_0 axis. Numerical integration shows that at $z_0 \sim bf$ $|\Psi(z_0)|$ is of the same character as in Section 3. Applicability of the PA near this focus is governed by condition

$$\kappa_f = k\left|S_2\right|x_{\max}^4\Big|_{z_0=bf} \sim k\alpha^3 a\,\frac{(b-1)}{b} \ll 1 \ . \tag{20}$$

5. DISCUSSIONS

One can separate out a circumstance common for the problems in Sections 3 and 4. Originally spherical (or approximately parabolic) wavefront experiences certain aberration that causes oscillations near focus which the PA is unable to trace. Since the origin of aberrations has no any qualitative significance on can state that the problems dealing with arbitrary (nonspherical) wave beams should be solved using more general than the PA methods. The values like those given by relations (19) and (20) will serve as quantitative criteria. However, some modificaions can also be used under other boundary conditions.

As follows from relations (19) and (20), inaccuracy of the PA increases with geometrical divergence (angular spectrum) of the external field that well agrees with the discussion in Ref. 4. At the same time data presented in Fig. 3 make one introduce a correction. Thus, e.g., the use of the PA is justified if the characteristic size a of a beam on the boundary surface and at focus ρ_0 differs not very much, i.e., (for example, for a Gaussian beam)

$$\eta = \rho_0/a \sim \left(\frac{2f}{ka}\right)\frac{1}{a} = \frac{2}{ka}\frac{1}{\alpha} \quad 1 \ . \tag{21}$$

In other words, since in Eq. (21) $(2/ka) \equiv \varphi$ is the diffractional divergence of the external beam the geometrical divergence must be not merely small, but close to the diffractional one.

It is interesting to note that, virtually, similar conclusion can be drawn from the necessary PA condition (13) itself. The latter can be presented in an equivalent (in the sense that the expansions of R and R_f will not differ from inequality (14)) form

$$\frac{|x|}{z_0} \sim \frac{|y|}{z_0} \sim \frac{|x_0|}{z_0} \sim \frac{|y_0|}{z_0} \sim \frac{|x|}{f} \sim \frac{|y|}{f} \sim \frac{a}{f} \ll 1 \ . \tag{22}$$

It follows from inequality (22) that the transition to the PA is quite faultless mathematically only under condition (21). If there appears strong focusing ($\eta \ll 1$), it is necessary to replace inequality (22) by

$$\frac{|x|}{z_0} \sim \frac{|y|}{z_0} \sim \frac{|x|}{f} \sim \frac{|y|}{f} \sim \frac{a}{f} \ll 1, \ \frac{|x_0|}{z_0} \sim \frac{|y_0|}{z_0} \sim \left(\frac{a}{f}\right)^j \ll 1 \ , \tag{23}$$

where j takes values from 1 (for $\alpha = \varphi$) up to, generally speaking, any large number depending on α and a from Eq. (21).

Using inequalities (23), it is easy to understand why the representation of S in the form of Eq. (17) becomes more accurate than Eq. (16), which results, when passing to the PA, in mathematical incorrectness leading to errors in final results. Really, assuming $\alpha = 10^{-1}$ for $a = 1$ cm, we can see from Eq. (21) that $\rho_0 \sim 0.2{\cdot}10^{-3}$ cm, and, consequently, j in inequalities (23) cannot be less than 4. This means that the expansion of R used in the PA

$$R \simeq z_0\left[1 + \frac{(x - x_0)^2 + (y - y_0)^2}{2\, z_0^2}\right]$$

contains terms $xx_0/z_0^2 \sim yy_0/z_0^2$ of the minus fifth order of magnitude according to inequalities (23). The error is evident, for the next term of the expansion (the same is valid for R_f, see inequalities (14)) of the type $(x^2 + y^2)/8z_0^4$ is of the minus fourth order of magnitude according to inequalities (23), i.e., the truncated terms exceed in value the remaining ones. Such an inaccuracy is the more essential the stronger the focusing and the larger the beam size is on the boundary. These conclusions are also evident from relations (19) and (20).

There is another quite curious point. When $m = 1$ the function S from Eq. (15) is represented in the PA in the form

$$S = R - R_f \simeq z_0 - f + \frac{(x - x_0)^2 + (y - y_0)^2}{2z_0} - \frac{x^2 + y^2}{2f}\ . \quad (24)$$

As a matter of fact, it follows from the above discussion that there were done two errors at a time, when writing Eq. (24), i.e., the series (14) for R and R_f were truncated. However, here these errors compensate each other $-\kappa_f = 0$ and Eq. (24) is not worse than $S = R - R_f$. But "making only one error" (no matter in R or in R_f), we shall compute Ψ less accurately. In this case κ_f will correspond to maximum value ($m \to \infty$) of the criteria (19) and Eq. (15) will give strong oscillations near focus, that must not occur in this case. This situation illustrates the statement from the end of Section 2.

6. CONCLUSION

This paper was aimed at only emphasizing qualitative difference between the exact and PA solutions as well as at pointing out the key items of the PA theory that can yield (in certain situations) to incorrect results. Just for this reason the quantitative aspects of concrete problems considered above remain to be open. The questions associated, for example, with the distribution of the field amplitude and phase over the beam cross section near focus have not been discussed at all. It was assumed that in any case the use of the PA must be preceded by an analysis of the PA applicability. Therefore, the PA can hardly be considered versatile.

In some controversial situations it is expedient to use either the exact solution (15) or the succeeding the PA approximation (17) in order to avoid, for certain, the errors in solving the problems of this kind. The method presented in Ref. 11 (field near a three–dimensional focus) used together with relations (23) seems to be mathematically faultless.

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